Give detailed explanations for your answers.
On in class exams I assign four problems. Each is worth 25 points.
I try to assign problems from different topics that we covered.
Below are several problems to help you get used to my style of exam questions.

1. Consider $2 \times 3$ matrix

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -4 & 1 \end{bmatrix}$$

and the linear transformation $T \mathbf{x} = A \mathbf{x}$ which is defined on $\mathbb{R}^3$ and with values in $\mathbb{R}^2$. Determine whether $T$ is injective (one-to-one). Justify your answer. If you claim that this transformation is not injective find two distinct vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$ such that $T \mathbf{x}_1 = T \mathbf{x}_2$.

Determine whether $T$ is surjective (onto). Justify your answer. If you claim that $T$ is surjective then for each $\mathbf{b}$ in $\mathbb{R}^2$ find a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that $T \mathbf{x} = \mathbf{b}$.

2. Consider $3 \times 4$ matrix

$$A = \begin{bmatrix} -1 & 2 & 3 & -6 \\ 2 & -4 & 1 & 5 \\ 1 & -2 & 2 & 1 \end{bmatrix}$$

and the linear transformation $T \mathbf{x} = A \mathbf{x}$ which is defined on $\mathbb{R}^4$ and with values in $\mathbb{R}^3$. Determine whether $T$ is injective (one-to-one). Justify your answer. Determine whether $T$ is surjective (onto). Justify your answer.

3. In this problem we assume that $n$ is an integer such that $n > 1$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$; that is $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices. Consider the following expressions

$$\mathbf{u}^\top \mathbf{v}, \quad \mathbf{v}^\top \mathbf{u}, \quad \mathbf{v} \mathbf{u}^\top, \quad \mathbf{u} \mathbf{v}^\top.$$

Here, the symbol $^\top$ denotes the transpose of a matrix. Give detailed answers to the following questions:

(a) Is it possible that $\mathbf{u}^\top \mathbf{v} = \mathbf{v}^\top \mathbf{u}$?

(b) Is it possible that $\mathbf{u}^\top \mathbf{v} = \mathbf{v} \mathbf{u}^\top$?

(c) Is it possible that $\mathbf{u} \mathbf{v}^\top = \mathbf{v} \mathbf{u}^\top$?

(d) Does the expression $(\mathbf{u} \mathbf{v}^\top)\mathbf{v}$ make sense? If this expression makes sense, which kind of matrix is it?

4. In this problem $A$ is $n \times n$ matrix. Without using the Invertible Matrix Theorem give explanations for the following implications.

(a) If the equation $A \mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then there is an $n \times n$ matrix $D$ such that $AD = I$. Explain why.

(b) If there is an $n \times n$ matrix $D$ such that $AD = I$, then the equation $A \mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why.

(c) If there is an $n \times n$ matrix $C$ such that $CA = I$, then the equation $A \mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why.
5. This problem is about invertible matrices. Let \( A \) be an \( n \times n \) matrix.

(a) State the definition of an invertible matrix.
(b) Prove the implication: If \( A \) is invertible, then \( A \) is row equivalent to \( I_n \).
(c) Prove the implication: If \( A \) is row equivalent to \( I_n \), then \( A \) is invertible.

6. For each matrix below determine whether it is invertible or not. Explain your claim. If a matrix is invertible find its inverse.

\[
A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},
\]
\[
D = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1/2 \\ -2 & 1 \end{bmatrix}.
\]

7. Consider the matrix \( A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \).

(a) Find \( A^{-1} \). Prove that your answer is correct by calculating \( AA^{-1} \).
(b) Use the inverse \( A^{-1} \) to find \( x_1, x_2, x_3 \) such that
\[
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}
\]

8. Consider the matrices \( A = \begin{bmatrix} 1 & -3 \\ -1 & k \end{bmatrix} \) and \( B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \).

(a) What values of \( k \) (if any) will make \( A \) invertible?
(b) What values of \( k \) (if any) will make \( AB \) invertible?
(c) What values of \( k \) (if any) will make \( AB = BA \)?

9. Let \( A \) be an unknown square matrix. To \( A \) we perform the following row operations:

- Row 1 gets replaced by \( \frac{1}{2} \) times Row 1. (\( R_1 \mapsto \frac{1}{2} R_1 \))
- Rows 2 and 3 are swapped (\( R_2 \mapsto R_3, R_3 \mapsto R_2 \))
- Row 2 gets replaced by \( -3 \) Row 2. (\( R_2 \mapsto -3 R_2 \))
- Row 3 gets replaced by Row 3 minus 6 Row 2. (\( R_3 \mapsto R_3 - 6 R_2 \))

The resulting matrix \( B \) has determinant \( \det B = 4 \). What is the determinant of the unknown matrix \( A \).

10. Let \( A \) be an unknown \( 3 \times 3 \) matrix. To \( A \) we perform the following row operations:
• Row 1 gets replaced by \( \frac{1}{2} \) times Row 1. \((R_1 \mapsto \frac{1}{2}R_1)\)
• Rows 2 and 3 are swapped \((R_2 \mapsto R_3, R_3 \mapsto R_2)\)
• Row 2 gets replaced by \(-3\) Row 2. \((R_2 \mapsto -3R_2)\)
• Row 3 gets replaced by Row 3 minus 6 Row 2. \((R_3 \mapsto R_3 - 6R_2)\)

The resulting matrix \(B\) is the identity matrix. What is the matrix \(A\)?

11. Prove that
\[
\begin{vmatrix}
1 & x & x^2 \\
1 & y & y^2 \\
1 & z & z^2
\end{vmatrix} = (y - x)(z - x)(z - y).
\]

12. Determine whether it is possible to write the matrix
\[
M = \begin{bmatrix}
1 & 0 & 1 \\
-2 & 1 & -2 \\
0 & 2 & 1
\end{bmatrix}
\]
as a product of elementary matrices. If you claim that it is possible to write \(M\) as a product of elementary matrices, then find elementary matrices whose product is \(M\). If you claim that it is not possible to write \(M\) as a product of elementary matrices, then justify your claim.

13. Calculate the determinant
\[
\begin{vmatrix}
1 & 2 & 0 & 0 \\
3 & 4 & 0 & 0 \\
5 & 0 & 6 & 7 \\
0 & 0 & 8 & 9
\end{vmatrix}.
\]

14. Determine \(h\) such that
\[
\begin{vmatrix}
1 & 2 & 3 & 0 \\
0 & 2 & 1 & h \\
2 & -2 & 1 & 0 \\
1 & 0 & 0 & 1
\end{vmatrix} = 0.
\]

15. (This problem has too many items for all of them to be on an exam.) Consider the \(3 \times 5\) matrix
\[
A = \begin{bmatrix}
-1 & 2 & 3 & 7 & -6 \\
-2 & 4 & 1 & 4 & -7 \\
1 & -2 & 2 & 3 & 1
\end{bmatrix}
\]

(a) Row reduce the matrix \(A\) to the reduced row echelon form.
(b) Celebrate your correct row reduction by multiplying the \(3 \times 2\) matrix which consists of the pivot columns of \(A\) by \(2 \times 5\) matrix which consists of nonzero rows of the reduced row echelon form of \(A\).
(c) Find a basis for \(\text{Nul} \ A\).
(d) Find a basis for \(\text{Col} \ A\).
(e) Express each nonpivot column of \(A\) as a linear combination of the basis for \(\text{Col} \ A\) that you found.
(f) Find a basis for \(\text{Row} \ A\).
(g) Express each row of \(A\) as a linear combination of the basis for \(\text{Row} \ A\) that you found.
16. The matrix $A$ and its reduced row echelon form are given below.

\[
A = \begin{bmatrix}
1 & 1 & 3 & 2 & 1 & 1 \\
1 & 2 & 5 & 3 & 0 & 3 \\
2 & 3 & 8 & 5 & 1 & 4 \\
2 & 2 & 6 & 4 & 3 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) Find the rank of $A$ and the dimension of the null space of $A$.

(b) Find the rank of $A^\top$ and the dimension of the null space of $A^\top$.

(c) Denote the columns of $A$ by $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$. Based on the given RREF of $A$, find a basis for the column space of $A$. Denote this basis by $\mathcal{A}$. Calculate the vector $[a_6]_{\mathcal{A}}$.

(d) Find a basis for the null space of $A$.

17. Given two bases $\mathcal{A}$ and $\mathcal{B}$ calculate the change of coordinates matrices $P_{\mathcal{A}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{A}}$. There are several examples on the class website. One is with a picture, one is with given vectors.