ON A ZERO OF A CONTINUOUS FUNCTION

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In this note a and b are real numbers and $a < b$.

Definition 1. A function $f : I \to \mathbb{R}$ is continuous at a point $z \in I$ if there exists a function $\delta_z : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $\epsilon > 0$ we have

$$x \in I, \quad |x - z| < \delta_z(\epsilon) \quad \Rightarrow \quad |f(x) - f(z)| < \epsilon.$$

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. If $f(a) > 0$ and $f(b) < 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.*

Proof. Assume $f(a) > 0$ and $f(b) < 0$. Set

$$W = \Big\{ w \in [a, b) : \quad f(x) > 0 \ \ \forall\, x \in [a, w] \ \Big\}.$$

Clearly $a \in W$ and $W \subset [a, b]$. Therefore, $c = \sup W$ exists by the Completeness Axiom. Since $a \in W$ and b is an upper bound for W we have $c \in [a, b]$.

1. Next we show that W does not have a maximum.
Let $v \in W$ be arbitrary. Then $v < b$ and $f(x) > 0$ for all $x \in [a, v]$. In particular, $f(v) > 0$. Since f is continuous at v, setting

$$\mu = \frac{1}{2} \min\big\{\delta_v\big(f(v)/2\big), b - v\big\} > 0$$

yields that $f(x) > 0$ for all $x \in [v, v + \mu]$. (Prove this as an exercise!) As, by definition of W, $f(x) > 0$ for all $x \in [a, v]$, we conclude that $f(x) > 0$ for all $x \in [a, v + \mu]$. Since clearly $v + \mu < b$, we have $v + \mu \in W$. As $v + \mu > v$, we proved that v is not a maximum of W. Thus, $c \notin W$. In particular $c > a$.

2. Here we show that $f(c) \geq 0$ and $c < b$.
Assume $y \in (a, b]$ and $f(y) < 0$. Since f is continuous at y, setting

$$\nu = \min\big\{\delta_y\big(-f(y)/2\big), y - a\big\} > 0$$

yields that $f(x) < 0$ for all $x \in (y - \nu, y]$. (Prove this as an exercise!) Therefore, $(y - \nu, y] \cap W = \emptyset$. Thus we proved:

$$y \in (a, b], \quad f(y) < 0 \quad \Rightarrow \quad \exists\, \nu > 0 \quad \text{such that} \quad (y - \nu, y] \cap W = \emptyset.$$

A contrapositive of this implication is

$$y \in (a, b], \quad \forall\, \epsilon > 0 \quad (y - \epsilon, y] \cap W \neq \emptyset \quad \Rightarrow \quad f(y) \geq 0.$$

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Since $c = \sup W$ has the properties

$$c \in (a, b] \quad \text{and} \quad \forall\, \epsilon > 0 \quad (c - \epsilon, c] \cap W \neq \emptyset,$$

we conclude that $f(c) \geq 0$. Since $f(b) < 0$, this implies that $c < b$.

3. Now we prove $[a, c) \subset W$.

Let $u \in [a, c)$ be arbitrary. Since $u < c$, u is not an upper bound of W. Therefore, there exists $v \in W$ such that $u < v < c$. Since $f(x) > 0$ for all $x \in [a, v]$, we have (trivially) $f(x) > 0$ for all $x \in [a, u]$. Thus $u \in W$ and therefore $[a, c) \subset W$.

4. Finally we prove $f(c) \leq 0$.

Assume that $v \in [a, b)$ and $[a, v) \subset W$. The following implication is obvious: If $f(v) > 0$, then $v \in W$. Its contrapositive is: If $v \notin W$, then $f(v) \leq 0$. Since we already proved that $c \in (a, b)$, $[a, c) \subset W$ and $c \notin W$, the last implication yields $f(c) \leq 0$.

In **2** we proved that $f(c) \geq 0$ and in **4** we proved that $f(c) \leq 0$. Therefore $f(c) = 0$. $\square$