Axioms for the Set $\mathbb{R}$ of Real Numbers

**Axiom 1** (AE: Addition exists). If $a, b \in \mathbb{R}$, then the sum of $a$ and $b$, denoted by $a + b$, is a uniquely defined number in $\mathbb{R}$.

**Axiom 2** (AA: Addition is associative). For all $a, b, c \in \mathbb{R}$ we have $a + (b + c) = (a + b) + c$.

**Axiom 3** (AC: Addition is commutative). For all $a, b \in \mathbb{R}$ we have $a + b = b + a$.

**Axiom 4** (AZ: Addition has 0). There is an element 0 in $\mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$.

**Axiom 5** (AO: Addition has opposites). If $a \in \mathbb{R}$, then the equation $a + x = 0$ has a solution $-a \in \mathbb{R}$. The number $-a$ is called the **opposite** of $a$.

**Axiom 6** (ME: Multiplication exists). If $a, b \in \mathbb{R}$, then the product of $a$ and $b$, denoted by $ab$, is a uniquely defined number in $\mathbb{R}$.

**Axiom 7** (MA: Multiplication is associative). For all $a, b, c \in \mathbb{R}$ we have $a(bc) = (ab)c$.

**Axiom 8** (MC: Multiplication is commutative). For all $a, b \in \mathbb{R}$ we have $ab = ba$.

**Axiom 9** (MO: Multiplication has 1). There is an element $1 \neq 0$ in $\mathbb{R}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$.

**Axiom 10** (MR: Multiplication has reciprocals). If $a \in \mathbb{R}$ is such that $a \neq 0$, then the equation $a \cdot x = 1$ has a solution $a^{-1} = \frac{1}{a}$ in $\mathbb{R}$. The number $a^{-1} = \frac{1}{a}$ is called the **reciprocal** of $a$.

**Axiom 11** (DL: Distributive law, the connection between addition and multiplication). For all $a, b, c \in \mathbb{R}$ we have $a(b + c) = ab + ac$.

**Axiom 12** (OE: Order exists). Given any $a, b \in \mathbb{R}$, exactly one of these statements is true: $a < b$, $a = b$, or $b < a$. (The symbol $a \leq b$ stands for $a < b$ or $a = b$.)

**Axiom 13** (OT: Order is transitive). Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

**Axiom 14** (OA: Order respects addition). Given any $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

**Axiom 15** (OM: Order respects multiplication). Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

**Axiom 16** (CA: Completeness Axiom). If $A$ and $B$ are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

The end of axioms.

All statements about real numbers that are studied in beginning mathematical analysis courses can be deduced from these sixteen axioms.

The formulation of the **Completeness Axiom** given as **Axiom 16** is not standard. This formulation I found in the book *Mathematical analysis* by Vladimir Zorich, published by Springer in 2004. Zorich’s formulation is easier to state and it is equivalent to the standard formulation of the Completeness Axiom.