ON THE REPRODUCING KERNEL OF A PONTRYAGIN SPACE OF VECTOR VALUED POLYNOMIALS

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Abstract. We give necessary and sufficient conditions under which the reproducing kernel of a Pontryagin space of $d \times 1$ vector polynomials is determined by a generalized Nevanlinna pair of $d \times d$ matrix polynomials.

1. Introduction

1.1. By Baire’s category theorem, the Pontryagin space $\mathcal{B}$ in the title is necessarily finite dimensional (see Remark 2.1 below) and hence is a reproducing kernel space. Indeed, if $(\mathcal{B}, \mathcal{B}^*)$ is an $n$-dimensional Pontryagin space of $d \times 1$ vector polynomials and if $B(z)$ is a $d \times n$ matrix polynomial whose columns $B_k(z)$, $k \in \{1, \ldots, n\}$, form a basis of $\mathcal{B}$, then the reproducing kernel of $\mathcal{B}$ is the $d \times d$ matrix polynomial in $z$ and $w^*$ given by

$$K(z, w) = B(z)G^{-1}B(w)^*, \quad z, w \in \mathbb{C},$$

where $G$ is the $n \times n$ Gram matrix associated with $B(z)$, that is,

$$G = [g_{jk}]_{j,k=1}^n, \quad g_{jk} = [B_k, B_j]_{\mathcal{B}}, \quad j, k \in \{1, \ldots, n\}$$

(see [3] Example 2.1.8 and the remark following it). The reproducing kernel of a reproducing kernel space is unique but can often be written in various ways. In this paper we give necessary and sufficient conditions under which $K(z, w)$ above is a polynomial Nevanlinna kernel. This means that it can be written in the form

$$K(z, w) = K_{M,N}(z, w) := \frac{M(z)N(w)^* - N(z)M(w)^*}{z - w^*}, \quad z, w \in \mathbb{C}, \quad z \neq w^*,$$

where $M(z)$ and $N(z)$ are $d \times d$ matrix polynomials such that

$$M(z)N(z^*)^* - N(z)M(z^*)^* = 0 \quad \text{for all} \quad z \in \mathbb{C}$$

and

$$\text{rank}[M(z) \quad N(z)] = d \quad \text{for at least one} \quad z \in \mathbb{C}. \quad (1.1)$$

If, in addition, the equality in $(1.1)$ holds for all $z \in \mathbb{C}$, then the Nevanlinna kernel $K_{M,N}(z, w)$ is called a full Nevanlinna kernel.

The following theorem is the main result in this paper. It is proved in Section 4.

Date: March 8, 2011.

2000 Mathematics Subject Classification. 46C20, 46E22, 47A06.

Key words and phrases. Polynomials, generalized Nevanlinna pair, Pontryagin space, reproducing kernel, Smith normal form, Forney indices, symmetric operator, defect number, self-adjoint extension, $Q$-function.
Theorem 1.1. Let $\mathcal{B}$ be a (finite dimensional) Pontryagin space of $d \times 1$ vector polynomials. Denote by $S_{\mathcal{B}}$ the operator of multiplication by the independent variable in $\mathcal{B}$ and by $E_{\alpha}$ the operator of evaluation at a point $\alpha \in \mathbb{C}$. Then the reproducing kernel of $\mathcal{B}$ is a polynomial Nevanlinna kernel if and only if the following two conditions hold:

(A) The operator $S_{\mathcal{B}}$ is symmetric in $\mathcal{B}$.

(B) For some $\alpha \in \mathbb{C}$ we have $\text{ran}(S_{\mathcal{B}} - \alpha) = \mathcal{B} \cap \text{ker} E_{\alpha}$.

Under these equivalent conditions, the reproducing Nevanlinna kernel is full if and only if the equality in (B) holds for all $\alpha \in \mathbb{C}$.

We think Theorem 1.1 is new, possibly even in the positive definite case, that is, the case where the space $\mathcal{B}$ is a reproducing kernel Hilbert space of vector polynomials. In that case $\mathcal{B}$ in the theorem is a special case of L. de Branges’ Hilbert spaces of entire functions. For scalar functions, see [12]; for vector functions, see [13] and [14]. In particular, [14] Theorems 1-3 are closely related to Theorem 1.1. For results on the indefinite scalar case we refer to the series of papers on Pontryagin spaces of entire functions by M. Kaltenbäck and H. Woracek. More specifically, [25, Theorem 5.3] is closely related to Theorem 1.1 with $\mathcal{B}$ in the above theorems is analogous to the so-called Szegő space, in the Hilbert space setting defined and studied in [33, 34] and in the Pontryagin space setting in [13]. In the literature there are many papers characterizing special forms of the reproducing kernel of a reproducing kernel space.
1.3. A pair \( \{M(z), N(z)\} \) of \( d \times d \) matrix functions \( M(z) \) and \( N(z) \) is called a generalized Nevanlinna pair if the functions are meromorphic on \( \mathbb{C} \setminus \mathbb{R} \), the intersection of the domains of holomorphy \( \text{hol}(M) \) of \( M(z) \) and \( \text{hol}(N) \) of \( N(z) \) is symmetric with respect to the real axis,

\[
M(z)N(z^*) - N(z)M(z^*) = 0 \quad \text{for all} \quad z \in \text{hol}(M) \cap \text{hol}(N),
\]

\[
\text{rank}[M(z)N(z)] = d \quad \text{for at least one} \quad z \in \text{hol}(M) \cap \text{hol}(N),
\]

and the Nevanlinna kernel

\[
K_{M,N}(z,w) := \frac{M(z)N(w^*) - N(z)M(w^*)}{z - w^*}, \quad z, w \in \text{hol}(M) \cap \text{hol}(N), z \neq w^*,
\]

has a finite number of negative squares. Here, by a finite number of negative squares we mean that the set of numbers of negative eigenvalues counted according to multiplicity of the self-adjoint matrices of the form

\[
[x_j^* K_{M,N}(z_j, z_i)x_i]_{i,j=1}^n
\]

has a maximum. If this maximum is \( \kappa \), then we say that the pair and the kernel have \( \kappa \) negative squares. If \( \kappa = 0 \) the adjective “generalized” is omitted; also in that case the matrix functions are holomorphic at least on \( \mathbb{C} \setminus \mathbb{R} \). The number of positive squares is defined in the same way. The pair and kernel are called full if the equality in (1.5) holds for all \( z \in \text{hol}(M) \cap \text{hol}(N) \). If a (generalized) Nevanlinna pair \( \{M(z), N(z)\} \) is such that \( N(z) = I_d \), the \( d \times d \) identity matrix, then it is identified with its first entry \( M(z) \) and \( M(z) \) is a (generalized) Nevanlinna function.

Nevanlinna pairs and generalized Nevanlinna pairs have been used in interpolation and moment problems (see \([29, 4, 5, 8]\), the description of generalized resolvents (see \([27]\) and in the theory of boundary value problems with eigenvalue dependent boundary conditions (see \([17, 18, 15]\) and \([9]\)). Theorem 1.1 arose in our study \([10]\) of an eigenvalue problem for an ordinary differential operator in a Hilbert space with boundary conditions which depend polynomially on the eigenvalue parameter. In that paper we linearize the original problem by extending the Hilbert space with a finite dimensional Pontryagin space of \( d \times 1 \) vector polynomials. This paper concerns the structure of such spaces.

1.4. The Nevanlinna pair in a Nevanlinna kernel is not unique (see the paragraph before Example 6.6) and if \( \{M(z), N(z)\} \) is a pair that determines the kernel, then the polynomial matrix \( N(z) \) may be such that \( \det N(z) = 0 \) for all \( z \in \mathbb{C} \). In Section 5 we prove that one can always choose the pair so that \( \det N(z) \neq 0 \) and
the rational generalized Nevanlinna matrix function $N(z)^{-1}M(z)$ is essentially a $Q$-
function of the symmetric operator $S_{\mathfrak{B}}$. We show that every self-adjoint extension
of $S_{\mathfrak{B}}$ with nonempty resolvent set gives rise to a reproducing Nevanlinna kernel
for the space $\mathfrak{B}$. The proof of Theorem 1.1 given in Section 4 is geometric, the
proof of the first if statement in Theorem 1.1 given in Section 5 is analytic. The
last two examples in Section 6, Example 6.5 and Example 6.6, also serve to show
that this analytic proof is constructive. In the first part of Section 6 we present
three corollaries of Theorem 1.1.

In Section 2 we fix the notation related to vector and matrix polynomials and
we recall the Smith normal form and the Forney indices of a matrix polynomial.
Moreover, we prove some lemmas on the structure of a degenerate subspace of
a finite dimensional Pontryagin space, the defect numbers of a simple symmetric
relation in such a space and on polynomial Hermitian kernels. Although most
proofs in this paper are based on methods from linear algebra, in the sequel we
assume that the reader is familiar with (i) Pontryagin spaces and (multi-valued)
operators on such spaces such as symmetric and self-adjoint relations (as in [23],
[19] and [11]), (ii) generalized Nevanlinna matrix functions (as in [29, 30]) and (iii)
reproducing kernel Pontryagin spaces (as in [6, Chapter 1] and [3, Chapter 7]).

The notion of a $Q$-function of a simple symmetric operator in a Pontryagin space
is recalled in Chapter 5.

Acknowledgements. The authors thank Prof. Marius van der Put for many
discussions about Theorem 3.4 and another proof of it and Prof. Daniël Alpay for
acquainting them with his unpublished note [1] which among other things lead to
Remark 2.1 and pointing out the references [33, 34].

2. Notation and basic objects

2.1. The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, real numbers
and complex numbers. For $d \in \mathbb{N}$ the vector space of all $d \times 1$ vectors is written as
$\mathbb{C}^d$ and $I_d$ stands for the $d \times d$ identity matrix. The $k$-th row of $I_d$ will be denoted
by $e_{d,k}$. For $k \in \{1, \ldots, n\}$ the subspace of $\mathbb{C}^d$ spanned by $e_{d,1}, \ldots, e_{d,k}$ will be
called a top coordinate subspace of $\mathbb{C}^d$; it will be denoted by $\mathbb{C}^d_k$. The corresponding
$d \times d$ projection matrix is denoted by $P_{d,k}$. We consider $\mathbb{C}^d_0 = \{0\}$ a top coordinate
subspace spanned by the empty set.

By $\mathbb{C}^d[z]$ we denote the vector space over $\mathbb{C}$ of all polynomials with coefficients
in $\mathbb{C}^d$. The space $\mathbb{C}^d$ is identified with the subspace of all constant polynomials in
$\mathbb{C}^d[z]$. If $d = 1$ we simply write $\mathbb{C}[z]$ and $\mathbb{C}$. For $f \in \mathbb{C}^d[z] \setminus \{0\}$ with
$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$
and for the zero polynomial $0$ we define
$$\deg f = \max \{k \in \{0, \ldots, n\} : a_k \neq 0\} \quad \text{and} \quad \deg 0 = -\infty.$$ 
Matrix polynomials are written as $B(z), M(z), N(z), \ldots$, that is, with their argument $z$; we use the bold face $\mathbf{P}(z), \mathbf{S}(z), \ldots$, for $d \times 2d$ matrix polynomials. Vector
polynomials are sometimes written with and sometimes without their argument.
The Fraktur alphabet $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{S}, \mathfrak{I}, \ldots$ is used to denote vector subspaces of $\mathbb{C}^d[z]$. One
exception to this is that $\mathfrak{L}$ will be used for a subspace of $\mathbb{C}^{2d}[z]$. An inner
product on $\mathfrak{B}$ is denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{B}}$. In a vector space, the symbol $\oplus$ denotes the
direct sum of subspaces.
Remark 2.1. A Banach space with a countable Hamel basis is separable and hence, by [31], it is finite dimensional. Since \( \{z^n : n \in \{0\} \cup \mathbb{N}\} \) is a countable Hamel basis of \( \mathbb{C}[z] \), the space \( \mathbb{C}^d[z] \) and all its subspaces also have countable Hamel bases. Therefore any Pontryagin subspace of \( \mathbb{C}^d[z] \) is finite dimensional. In spite of this fact, to emphasize the finite dimensionality, we continue to speak of finite dimensional Pontryagin subspaces of \( \mathbb{C}^d[z] \).

We introduce some special subspaces of \( \mathbb{C}^d[z] \). Let \( n \in \{0\} \cup \mathbb{N} \). The symbol \( \mathbb{C}^d[z]_{<n} \) stands for the set of all \( f \in \mathbb{C}^d[z] \) such that \( \deg f < n \). In particular, \( \mathbb{C}^d[z]_{<1} = \mathbb{C}^d \) and \( \mathbb{C}^d[z]_{<0} = \{0\} \). A subspace \( \mathcal{E} \) of \( \mathbb{C}^d[z] \) is called canonical if there exist nonnegative integers \( \mu_k, k \in \{1, \ldots, d\} \), such that

\[
\mathcal{E} = \bigoplus_{k=1}^d (\mathbb{C}[z]_{<\mu_k})_{e_{d,k}}
\]

\[
= \{[p_1(z) \cdots p_d(z)]^\top : p_k(z) \in \mathbb{C}[z], \ deg p_k < \mu_k, k \in \{1, \ldots, d\}\}.
\]

The numbers \( \mu_1, \ldots, \mu_d \) will be called the degrees of \( \mathcal{E} \). Without loss of generality we can assume that they are ordered: \( \mu_1 \geq \cdots \geq \mu_d \geq 0 \). Then a canonical subspace is uniquely determined by its degrees. Clearly, the dimension of \( \mathcal{E} \) is the sum of its degrees.

Next we introduce some useful operators on \( \mathbb{C}^d[z] \). By \( P_{d,k}, k \in \{1, \ldots, d\} \), we denote the natural extension of \( P_{d,k} \) to \( \mathbb{C}^d[z] \), by \( S : \mathbb{C}^d[z] \to \mathbb{C}^d[z] \) the operator of multiplication by the independent variable, that is,

\[
(Sf)(z) = zf(z), \quad f \in \mathbb{C}^d[z],
\]

and by \( E_\alpha : \mathbb{C}^d[z] \to \mathbb{C}^d \) the evaluation operator at the point \( \alpha \in \mathbb{C} \):

\[
E_\alpha(f) = f(\alpha), \quad f \in \mathbb{C}^d[z].
\]

It follows from the fundamental theorem of algebra that

\[
(2.1) \quad \text{ran}(S - \alpha) = \ker E_\alpha.
\]

A wide class of operators on \( \mathbb{C}^d[z] \) is induced by \( d \times d \) matrix polynomials. If \( M(z) \) is such a polynomial we define the operator \( M : \mathbb{C}^d[z] \to \mathbb{C}^d[z] \) by

\[
(Mf)(z) = M(z)f(z), \quad f \in \mathbb{C}^d[z].
\]

Clearly, \( MS = SM \). A square matrix polynomial is unimodular if its determinant is identically equal to a nonzero constant. If \( M(z) \) is a unimodular matrix polynomial we will call \( M \) a unimodular operator. In this case \( M \) is a bijection and its inverse is also a unimodular operator.

2.2. In the sequel we use that any nonzero \( d \times n \) matrix polynomial \( B(z) \) admits a Smith normal form representation (see for example [22, Satz 6.3] or [21]):

\[
(2.2) \quad B(z) = U(z) \begin{bmatrix} D(z) & 0 \\ 0 & 0 \end{bmatrix} V(z),
\]

where \( U(z) \) is a \( d \times d \) unimodular matrix polynomial, \( V(z) \) is an \( n \times n \) unimodular matrix polynomial and the matrix in the middle is a \( d \times n \) matrix in which, for some \( l \in \{1, \ldots, \min\{d, n\}\} \), \( D(z) \) is a diagonal \( l \times l \) matrix polynomial with monic diagonal entries: \( D(z) = \text{diag}(b_1(z), \ldots, b_l(z)) \) such that \( b_i(z) \) is divisible by \( b_{i+1}(z), i \in \{1, \ldots, l-1\} \). Notice that \( \text{rank} B(\alpha) = l \) if and only if \( b_l(\alpha) \neq 0 \). If
for some \( z \in \mathbb{C} \) the rank of \( B(z) \) is \( d \) (or \( n \), respectively), then \( l = d \) (or \( l = n \)) and the zero block row (column) in the matrix in the middle of the right hand side in (2.2) is not present.

**Remark 2.2.** The matrix in the middle of the right hand side in (2.2) is uniquely determined by \( B(z) \). In this paper \( B(z) \) often is a matrix polynomial whose columns form a basis of a subspace \( \mathfrak{B} \) of \( \mathbb{C}^d[z] \). Then for any \( d \times n \) matrix polynomial \( B_1(z) \) whose columns also form a basis of \( \mathfrak{B} \), the middle term of its Smith normal form is identical to that of \( B(z) \). Thus, the number \( l \) and the monic polynomials \( b_j(z), j \in \{1, \ldots, l\} \), above are uniquely determined by the subspace \( \mathfrak{B} \) of \( \mathbb{C}^d[z] \).

2.3. Let \( S(z) \) be a \( d \times 2d \) polynomial matrix. For \( j \in \{1, \ldots, d\} \) let \( \sigma_j \) be the degree of the \( j \)-th row of \( S(z) \) (by definition, the degree of the transpose of the row). Define \( S_\infty \), the *internal degree* and the *external degree* of \( S(z) \) by:

\[
S_\infty = \lim_{z \to \infty} \begin{bmatrix} z^{-\sigma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z^{-\sigma_d} \end{bmatrix} S(z),
\]

\[
\text{extdeg } S(z) = \sigma_1 + \cdots + \sigma_d, \quad \text{and}
\]

\[
\text{intdeg } S(z) = \{ \deg m(z) : m(z) \text{ is a } d \times d \text{ minor of } S(z) \}.
\]

For a proof of the following theorem we refer to [32].

**Theorem 2.3.** Let \( P(z) \) be a \( d \times 2d \) matrix polynomial with rank \( P(z) = d \) for all \( z \in \mathbb{C} \). Let \( S(z) \) be a matrix polynomial in the family

\[
\{ U(z)P(z) : U(z) \text{ unimodular} \}.
\]

The following statements are equivalent:

(a) \( \text{extdeg } S(z) = \min \{ \text{extdeg } U(z)P(z) : U(z) \text{ unimodular} \} \).

(b) \( \text{rank } S_\infty = d \).

(c) \( \text{extdeg } S(z) = \text{intdeg } S(z) \).

(d) \( S(z)^* \) has the "predictable degree property":

For every \( u(z) = [u_1(z) \cdots u_d(z)]^T \in \mathbb{C}^d[z] \) we have

\[
\deg (S(z)^* u(z)) = \max \{ \sigma_j + \deg u_j(z), j \in \{1, \ldots, d\} \}.
\]

A matrix polynomial \( S(z) \) in the family (2.3) satisfying the conditions (a)–(d) is called *row reduced*. The multiset \( \{ \sigma_1, \ldots, \sigma_d \} \) of row degrees for each row reduced matrix in the family (2.3) is the same. Its elements are called the *Forney indices* of any of the matrices in the family (2.3), in particular of \( P(z) \). We extend this definition to the case where the \( d \times 2d \) matrix polynomial \( P(z) \) has full rank for some \( z \in \mathbb{C} \). For that we use the following lemma which is a standard tool in system theory, see for example [21].

**Lemma 2.4.** Let \( P(z) \) be a \( d \times 2d \) matrix polynomial with rank \( P(z) = d \) for some \( z \in \mathbb{C} \). Then \( P(z) \) admits the factorization:

\[
P(z) = G(z)T(z) \quad \text{for all } z \in \mathbb{C},
\]

where \( G(z) \) is a \( d \times d \) matrix polynomial with \( \det G(z) \neq 0 \) and \( T(z) \) is a \( d \times 2d \) matrix polynomial with rank \( T(z) = d \) for all \( z \in \mathbb{C} \). This factorization is essentially unique, meaning that if also \( P(z) = G_1(z)T_1(z) \) for all \( z \in \mathbb{C} \), where \( G_1(z) \) and
The Forney indices of $P(z)$ in the lemma are by definition the Forney indices of the matrix polynomial $T(z)$ in the factorization (2.4). By the second part of the lemma, this definition is independent of the choice of the matrix $G(z)$ in this factorization.

For convenience of the reader we give a proof of Lemma 2.4 based on the Smith normal form of a matrix polynomial.

Proof of Lemma 2.4. Let $P(z)$ have the Smith normal form (2.2). The assumptions imply that $l = d$ and that the matrix in the middle of (2.2) is equal to $[D(z) \ 0]$. Set $G(z) = U(z)D(z)$ and $T(z) = [I_d \ 0] V(z)$. Then the factorization (2.4) holds and $G(z)$ and $T(z)$ have the properties mentioned in the the lemma. To prove uniqueness we use the fact that, since $T(z)$ and $T_1(z)$ have full rank for all $z \in \mathbb{C}$, they have right inverses, see [24]. These are $2d \times d$ matrix polynomials $S(z)$ and $S_1(z)$ such that $T(z)S(z) = I_d$ and $T_1(z)S_1(z) = I_d$ for all $z \in \mathbb{C}$. Define the matrix polynomials $E(z) = T_1(z)S(z)$ and $F(z) = T(z)S_1(z)$. Then the equality $G(z)T(z) = G_1(z)T_1(z)$ implies $E(z) = G_1(z)^{-1} G(z)$ and $F(z) = G(z)^{-1} G_1(z)$, hence $E(z)F(z) = I_d$ for all but finitely many $z \in \mathbb{C}$. By continuity the last equality holds for all $z \in \mathbb{C}$, hence $E(z)$ is unimodular and has the stated properties.

2.4. The next two lemmas concern finite dimensional Pontryagin spaces. By the positive (negative) index of a Pontryagin space $\mathcal{K}$ we mean the dimension of a maximal positive (negative) subspace of $\mathcal{K}$; evidently, the dimension of $\mathcal{K}$ is equal to the sum of the indices.

Lemma 2.5. Let $\mathcal{K}$ be a Pontryagin space with positive and negative index equal to $n$. Let $\mathcal{L}$ be a subspace of $\mathcal{K}$ with $\dim \mathcal{L} = 2n - \tau$. If $\mathcal{L}$ contains a maximal neutral subspace of $\mathcal{K}$, then $\mathcal{L}^\perp$ is the isotropic part of $\mathcal{L}$ and $\mathcal{L}/\mathcal{L}^\perp$ is a Pontryagin space with positive and negative index equal to $n - \tau$.

Proof. Let $\mathcal{N}$ be a maximal neutral subspace contained in $\mathcal{L}$. Since $\mathcal{N}^\perp = \mathcal{N}$, the inclusion $\mathcal{N} \subseteq \mathcal{L}$, yields $\mathcal{L}^\perp \subset \mathcal{N} \subset \mathcal{L}$. Therefore, $\mathcal{L}^\perp$ is the isotropic part of $\mathcal{L}$ and $\dim \mathcal{L}^\perp = \tau$. Let $\mathcal{L} = \mathcal{L}^\perp + \mathcal{L}_- + \mathcal{L}_+$ be a pseudo-fundamental decomposition of $\mathcal{L}$. Since $\mathcal{N}$ is a neutral subspace of $\mathcal{L}$, we have $n = \dim \mathcal{N} \leq \tau + \dim \mathcal{L}_\pm$. Therefore

$$2n - \tau = \dim \mathcal{L} = \tau + \dim \mathcal{L}_- + \dim \mathcal{L}_+ \geq \tau + n - \tau + n - \tau = 2n - \tau.$$  

This proves that $\dim \mathcal{L}_- = \dim \mathcal{L}_+ = n - \tau$. □

Recall that a symmetric relation $S$ in a Pontryagin space $\mathcal{K}$ is simple if $S$ has no non-real eigenvalues and $\mathcal{K} = \text{span} \{ \ker (S^* - z) : z \in \mathbb{C} \setminus \mathbb{R} \}$. Below mul $S^*$ stands for the multi-valued part of the adjoint $S^*$ of $S$: \text{mul} $S^* = \{ g \in \mathcal{K} : \{ 0, g \} \in S^* \}$.

Lemma 2.6. Let $S$ be a simple symmetric relation in a finite dimensional Pontryagin space of dimension $n$. Then the spaces $\text{mul} S^*$, $\ker S^*$, and $S^* \cap z I$, $z \in \mathbb{C}$, have the same dimension $d'$, say. In particular, the defect numbers of $S$ are both equal to $d'$. Furthermore, $\dim S = n - d'$ and $\dim S^* = n + d'$.

Proof. First notice that by [24] Proposition 2.4) $S$ is an operator and $S$ has no eigenvalues. The following statements are equivalent:

$T_1(z)$ have the same properties as $G(z)$ and $T(z)$, then for some unimodular $d \times d$ matrix polynomial $E(z): G_1(z) = G(z)E(z)^{-1}$ and $T_1(z) = E(z)T(z)$, $z \in \mathbb{C}$.  


(a) \( \dim(\text{mul} \, S^*) = d' \).
(b) \( \codim(\text{dom} \, S) = d' \).
(c) \( \codim(\text{ran} (S - z^*)) = d' \) for all \( z \in \mathbb{C} \).
(d) \( \dim(S^* \cap zI) = d' \) for all \( z \in \mathbb{C} \).

The relation \( (\text{dom} \, S)^\perp = \text{mul} \, S^* \) implies the equivalence (a) \( \iff \) (b). The equivalence (b) \( \iff \) (c) follows from the fact that \( S - z^* \) is one-to-one. By taking the orthogonal complements we obtain the equivalence (c) \( \iff \) (d). Notice that (d) with \( z = 0 \) implies that \( d' = \dim(\ker \, S^*) \). The equality \( \dim S = n - d' \) follows from (d) and the fact that \( S \) is an operator. Since \( \dim S^* = 2n - \dim S \) the last equality follows. \( \square \)

2.5. A \( d \times d \) matrix function \( K(z, w) \) will be called a polynomial Hermitian kernel if it is a polynomial of two variables \( z \) and \( w^* \) and \( K(z, w)^* = K(w, z) \), \( z, w \in \mathbb{C} \). This implies that the degree of \( K(z, w) \) as a polynomial in \( z \) equals the degree of \( K(z, w) \) as a polynomial in \( w^* \). If we denote this common degree by \( p - 1 \), then \( K(z, w) \) can be expanded as

\[
K(z, w) = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} A_{jk} z^j w^* \quad z, w \in \mathbb{C},
\]

where \( A_{jk}, j, k \in \{0, \ldots, p-1\} \), are \( d \times d \) matrices. Since \( K(z, w) \) is a Hermitian kernel the \( dp \times dp \) block matrix

\[
A = \begin{bmatrix}
A_{00} & \cdots & A_{0,p-1} \\
\vdots & \ddots & \vdots \\
A_{p-1,0} & \cdots & A_{p-1,p-1}
\end{bmatrix}
\]

is self-adjoint. It also follows that the number of negative squares of \( K(z, w) \) equals the number of negative eigenvalues of \( A \) and the number of positive squares of \( K(z, w) \) equals the number of positive eigenvalues of \( A \). The dimension of the reproducing kernel space corresponding to \( K(z, w) \) is the rank of \( A \). These observations are used in the proof of the following lemma.

**Lemma 2.7.** Let \( K(z, w) \) be a \( d \times d \) matrix polynomial Hermitian kernel of degree \( p - 1 \). For \( q \in \mathbb{N} \) set

\[
L_q(z, w) = i(z^q - w^q)K(z, w), \quad z, w \in \mathbb{C}.
\]

If \( q \geq p \), then the positive and the negative index of the reproducing kernel Pontryagin space with kernel \( L_q(z, w) \) are equal and coincide with the dimension of the reproducing kernel Pontryagin space with kernel \( K(z, w) \).

**Proof.** Write \( K(z, w) \) in the form (2.6) and denote by \( A \) the matrix (2.6). We calculate the coefficients of the matrix polynomial \( L_q(z, w) \) for \( q \geq p \):

\[
L_q(z, w) = i^q \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} A_{jk} z^j w^* \quad i \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} A_{jk} z^j w^*
\]

\[
= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} iA_{jk} z^{q+j} w^* + \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} (-i)A_{jk} z^j w^{*q+k}
\]

\[
= \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} iA_{(j-q)k} z^j w^* + \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} (-i)A_{(j-k)q} z^j w^*
\]

\[=0\]
where the 0 in the middle is a $d$ polynomial Nevanlinna kernel if it can be represented as

$$A = \sum_{j=0}^{q+p-1} \sum_{k=0}^{q+p-1} (iA_{j-k}) z^j w^k,$$

where we set $A_{jk} = 0$ whenever $j < 0$ or $k < 0$ or $j > p - 1$ or $k > p - 1$. In other words, the $2d(p+q) \times 2d(p+q)$ self-adjoint matrix formed by the coefficients of $L_q(z, w)$ is given by

$$B = \begin{bmatrix}
0 & 0 & -iA \\
0 & 0 & 0 \\
iA & 0 & 0
\end{bmatrix}$$

where the 0 in the middle is a $d(q-p) \times d(q-p)$ matrix. With

$$E = \frac{1}{\sqrt{2}} \begin{bmatrix}
I_{dp} & 0 & iI_{dp} \\
0 & I_{d(q-p)} & 0 \\
iI_{dp} & 0 & I_{dp}
\end{bmatrix}$$

we have $EE^* = I_{d(q+p)}$ and

$$E^* BE = E^* \begin{bmatrix}
0 & 0 & -iA \\
0 & 0 & 0 \\
iA & 0 & 0
\end{bmatrix} E = \begin{bmatrix}
A & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -A
\end{bmatrix}.$$

Therefore the rank of $B$ is twice the rank of $A$ and $B$ has equal numbers of positive and negative eigenvalues. Since the positive and negative index of the reproducing kernel Pontryagin space with kernel $L_q(z, w)$ coincide with the number of positive and negative eigenvalues of $B$ the lemma is proved. \hfill \Box

A polynomial reproducing Nevanlinna kernel introduced in the Introduction is a polynomial hermitian kernel. Since in the proof of Theorem 1.1 the polynomials in a Nevanlinna pair never appear separate we adopt the following equivalent definition of a polynomial Nevanlinna kernel: A $d \times d$ matrix function $K(z, w)$ is called a polynomial Nevanlinna kernel if it can be represented as

$$P(z)Q^{-1}P(w)^* = i(z - w^*)K(z, w) \quad \text{for all} \quad z, w \in \mathbb{C},$$

where $Q$ is a $2d \times 2d$ self-adjoint matrix with $d$ positive and $d$ negative eigenvalues and $P(z)$ is a $d \times 2d$ matrix polynomial such that $P(z)$ has rank $d$ for some $z \in \mathbb{C}$. With

$$Q = Q_1 := \begin{bmatrix}
0 & -iI_d \\
iI_d & 0
\end{bmatrix} \quad \text{and} \quad P(z) = \begin{bmatrix} M(z) & N(z) \end{bmatrix}$$

the definition from the Introduction is obtained from the new one. The assumptions on $Q$ imply that there exists a constant invertible matrix $T$ such that $Q = TQ_1 T^*$. Now, if we write $P(z)T = \begin{bmatrix} M(z) & N(z) \end{bmatrix}$, we have $K(z, w) = K_{M,N}(z, w)$. Since $P(z)$ is a polynomial, the condition that rank $P(z) = d$ for some $z \in \mathbb{C}$ implies that rank $P(z) = d$ for all but finitely many $z \in \mathbb{C}$. A polynomial Nevanlinna kernel will be called a full Nevanlinna kernel if $P(z)$ can be chosen such that rank $P(z) = d$ for all $z \in \mathbb{C}$.

3. Proof of Theorem 1.2

3.1. Let $\mathcal{B}$ be a vector subspace of $\mathbb{C}[^d[z]$. By $S_{\mathcal{B}}$ we denote the range restriction of $S$ to $\mathcal{B}$, that is,

$$\text{dom } S_{\mathcal{B}} = \mathcal{B} \cap S^{-1} \mathcal{B}, \quad (S_{\mathcal{B}} f)(z) = zf(z), \quad f \in \text{dom } S_{\mathcal{B}}.$$
In graph notation this means:

\[ S_\mathcal{B} = \{ (f, g) : f, g \in \mathcal{B}, g(z) = z f(z) \text{ for all } z \in \mathbb{C} \} \]

By (2.1), for \( \alpha \in \mathbb{C} \) we have

\[ \text{ran}(S_\mathcal{B} - \alpha) = (S - \alpha)(\mathcal{B} \cap S^{-1}\mathcal{B}) \subseteq \text{ran}(S - \alpha) \cap \mathcal{B} = \mathcal{B} \cap \ker E_\alpha. \]

The reverse inclusion is equivalent to the implication

\[ f \in \mathcal{B}, \ \alpha \in \mathbb{C}, \ f(\alpha) = 0 \ \Rightarrow \ f(z) = (z - \alpha)g(z) \ \text{ for some } \ g \in \text{dom}S_\mathcal{B}. \]

In some cases this implication does not hold. For example, it does not hold for any \( \alpha \in \mathbb{C} \) in the space \( \mathcal{B} \subseteq \mathbb{C}^2[z] \) given by

\[ \mathcal{B} = \left\{ \begin{bmatrix} a_0 + a_2 z^2 \\ b_0 + b_1 z \end{bmatrix} : a_0, a_2, b_0, b_1 \in \mathbb{C} \right\}. \]

Indeed, \( \mathcal{B} \) contains \([z^2 - \alpha^2 \ z - \alpha]^T\) which is 0 at \( z = \alpha \), but \( \mathcal{B} \) does not contain \([z + \alpha \ \ 1]^T\). That the implication, or equivalently, equality in (3.1), holds, is characterized in terms of canonical subspaces of \( \mathbb{C}^d[z] \) in Theorem 3.4 in the Introduction. This section is devoted to the proof of this theorem.

Let \( B(z) \) be a \( d \times n \) matrix polynomial whose columns form a basis for \( \mathcal{B} \), \( n = \dim \mathcal{B} \). Then, as will be shown in the proof of Theorem 1.2, the sets in (1.3) are equal to \( \{ \alpha \in \mathbb{C} : b_1(\alpha) \neq 0 \} \), where \( b_1(z) \) is the scalar polynomial in the Smith normal form (2.2) of \( B(z) \). We will first prove Theorem 1.2 for the case where the sets in (1.3) are equal to \( \mathbb{C} \), see Theorem 3.4 below. In this case \( W(z) \) is unimodular. The proof of Theorem 3.4 is based on the following three lemmas.

**Lemma 3.1.** Let \( \mathcal{B} \) be a finite dimensional subspace of \( \mathbb{C}^d[z] \) such that

\[ \text{ran}(S_\mathcal{B} - \alpha) = \mathcal{B} \cap \ker E_\alpha \text{ for all } \alpha \in \mathbb{C}. \]

If \( \text{dom}S_\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{B} \), then \( \text{ran}(S_\mathcal{B} - \alpha) = \mathcal{B}' \cap \ker E_\alpha \text{ for all } \alpha \in \mathbb{C} \).

**Proof.** Let \( f \in \mathcal{B}' \cap \ker E_\alpha \). Then \( f \in \mathcal{B} \cap \ker E_\alpha = \text{ran}(S_\mathcal{B} - \alpha) \), that is, \( f = Sg - \alpha g \) for some \( g \in \text{dom}S_\mathcal{B} \subseteq \mathcal{B}' \). From \( f, g \in \mathcal{B}' \) we infer \( g, Sg \in \mathcal{B}' \). Hence \( g \in \text{dom}S_\mathcal{B} \) and \( f = (S_\mathcal{B} - \alpha)g \). This proves \( \mathcal{B}' \cap \ker E_\alpha \subseteq \text{ran}(S_\mathcal{B} - \alpha) \). Since the reverse inclusion is obvious, the lemma is proved.

**Lemma 3.2.** Let \( \mathcal{B} \) be an \( n \)-dimensional subspace of \( \mathbb{C}^d[z] \). Then

\[ \mathcal{B} \cap \ker E_\alpha = \{0\} \text{ for all } \alpha \in \mathbb{C} \]

if and only if there exists a unimodular operator \( W \) such that \( \mathcal{B} = WC_n^d \), where \( C_n^d \) is a top coordinate subspace of \( \mathbb{C}^d \).

**Proof.** If \( n = 0 \), the statements are trivial with \( W(z) = I_d \). From now on we assume \( n \geq 1 \). If \( B(z) \) is any \( d \times n \) matrix polynomial whose columns form a basis of \( \mathcal{B} \), then, clearly,

\[ \{ \alpha \in \mathbb{C} : \text{rank} \ B(\alpha) = n \} = \{ \alpha \in \mathbb{C} : \mathcal{B} \cap \ker E_\alpha = \{0\} \}. \]

Assume (3.2). Let \( B(z) \) be a \( d \times n \) matrix polynomial whose columns form a basis of \( \mathcal{B} \). By (3.3), for all \( \alpha \in \mathbb{C} \) the rank of \( B(\alpha) \) is \( n \) and \( n \leq d \). Hence \( B(z) \)
admits the Smith normal form (see [2.2]): \(B(z) = U(z) \begin{bmatrix} I_n & 0 \end{bmatrix}^T V(z),\) where \(U(z)\) and \(V(z)\) are unimodular. Define

\[
W(z) = U(z) \begin{bmatrix} V(z) & 0 \\ 0 & I_{d-n} \end{bmatrix}.
\]

Then \(W(z)\) is a unimodular \(d \times d\) matrix polynomial and from \(B(z) = W(z) \begin{bmatrix} I_n & 0 \end{bmatrix}^T\) it follows that \(\mathcal{B} = WC_n^d\). This proves the only if statement.

To prove the if statement, assume that there exists a \(d \times d\) unimodular matrix polynomial \(W(z)\) such that \(\mathcal{B} = WC_n^d\), where \(C_n^d\) is a top coordinate subspace of \(C^d\). Then the columns of \(B(z) = W(z) \begin{bmatrix} I_n & 0 \end{bmatrix}^T\) form a basis of \(\mathcal{B}\) and the rank of \(B(\alpha)\) is \(n\) for all \(\alpha \in \mathbb{C}\). The equality \((3.2)\) follows from \((3.2)\). \(\square\)

**Lemma 3.3.** Let \(\mathcal{B}\) be an \(n\)-dimensional subspace of \(C^d[z]\) and let \(\mathcal{E}\) be a canonical subspace of \(C^d[z]\) with degrees \(\mu_1 \geq \cdots \geq \mu_d \geq 0\) of which \(k\) are positive. Assume \(\mathcal{E} + SC\) \(\subseteq \mathcal{B}\) and

\[
\mathcal{B} \cap \ker E_0 \subseteq \mathcal{E} + SC\quad \text{for all } \alpha \in \mathbb{C}.
\]

Then there exists a unimodular operator \(W\) which acts as an identity on \(\mathcal{E} + SC\) and is such that

\[
\mathcal{B} = W(C_m^d + \mathcal{E} + SC),
\]

where \(m = n - (\mu_1 + \cdots + \mu_d)\geq 0\).

Notice that \(C_m^d + \mathcal{E} + SC\) is a canonical subspace. If \(m \leq k\), then \(C_m^d + \mathcal{E} + SC\) coincides with \(\mathcal{E} + SC\) and \(W = I_d\).

**Proof.** If \(\mathcal{E} = \{0\}\) the statement follows from Lemma 3.2. From now on we assume \(\mathcal{E} \neq \{0\}\). Then \(\mu_1 > 0\), consequently \(k \in \{1, \ldots, d\}\) and \(C^d_k \subseteq \mathcal{E}\). We consider two cases: \(k = d\) and \(k < d\).

(i) Assume \(k = d\). Then \(C^d \subseteq \mathcal{E} \subseteq \mathcal{B}\). Let \(f \in \mathcal{B}\). It can be written as \(f(z) = f(0) + zh(z) = f(0) + (Sh)(z)\). Then \(Sh = f - f(0) \in \mathcal{B}\). Since \((Sh)(0) = 0\), by (3.3) we get \(Sh = \mathcal{B} \cap \ker E_0 \subseteq \mathcal{E} + SC\), which implies \(f = f(0) + Sh \in \mathcal{E} + SC\). That is, \(\mathcal{B} = \mathcal{E} + SC\). In this case \(m = d\) and with \(W = I_d\) the lemma is proved.

(ii) Assume \(k < d\). If \(\mathcal{E} + SC = \mathcal{B}\) then (3.6) holds with \(W = I_d\) and \(m = k\), implying that \(C^d_m \subseteq \mathcal{E}\). From now on we assume that \(\mathcal{E} + SC\) is a proper subspace of \(\mathcal{B}\). Recall that \(P_{d,k}\) is the coordinate projection. A trivial, but important observation is

\[
E_\alpha(\mathcal{E} + SC) = C^d_k = \text{ran} \ P_{d,k}\quad \text{for all } \alpha \in \mathbb{C}.
\]

Let \(\alpha \in \mathcal{E}\) be arbitrary and let \(f \in \mathcal{B}\) be such that \((I_d - P_{d,k})f(\alpha) = 0\). By (3.6), there exists a \(p \in \mathcal{E} + SC\) such that \(p(\alpha) = P_{d,k}f(\alpha)\), hence

\[
(f - p)(\alpha) = (I_d - P_{d,k})f(\alpha) + P_{d,k}f(\alpha) - p(\alpha) = 0,
\]

that is, \(f - p \in \ker E_\alpha\). Since also \(f - p \in \mathcal{B}\), (3.6) implies \(f - p \in \mathcal{E} + SC\). Thus both \(p\) and \(f - p\) belong to \(\mathcal{E} + SC\), implying that \(f \in \mathcal{E} + SC\). We have proved the implication:

\[
f \in \mathcal{B}, \quad \alpha \in \mathbb{C}\quad \text{and} \quad (I_d - P_{d,k})f(\alpha) = 0 \quad \Rightarrow \quad f \in \mathcal{E} + SC.
\]

Let \(\mathcal{L}_0\) be a subspace of \(\mathcal{B}\) be such that

\[
(\mathcal{E} + SC) \cap \mathcal{L}_0 = \{0\}\quad \text{and} \quad \mathcal{B} = (\mathcal{E} + SC) + \mathcal{L}_0.
\]
The dimension of $\mathfrak{L}_0$ is

$$j = n - (\mu_1 + \cdots + \mu_d + k) \geq 1.$$ 

Let $B_0(z)$ be a $d \times j$ matrix polynomial whose columns form a basis of $\mathfrak{L}_0$. Decompose $B_0(z)$ as

$$B_0(z) = \begin{bmatrix} B_{0,t}(z) \\ B_{0,b}(z) \end{bmatrix},$$

where $B_{0,t}(z)$ is a $k \times j$ matrix polynomial and $B_{0,b}(z)$ a $(d - k) \times j$ matrix polynomial. We will prove that

$$\text{rank } B_{0,b}(\alpha) = \text{rank}(I_d - P_{d,k}) B_0(\alpha) = j \text{ for all } \alpha \in \mathbb{C}. \quad (3.9)$$

The first equality is trivial. To prove the second let $\alpha \in \mathbb{C}$ be arbitrary and $x \in \mathbb{C}^j$ be such that $(I_d - P_{d,k}) B_0(\alpha) x = 0$. Set $f(z) = B_0(z) x$. Then $f \in \mathfrak{L}_0$ and $(I_d - P_{d,k}) f(\alpha) = 0$. By (3.8), $f \in (\mathfrak{C} + S\mathfrak{C}) \cap \mathfrak{L}_0$, consequently $f = 0$, that is, $B_0(\alpha) x = 0$ for all $x \in \mathbb{C}$. Since the columns of $B_0(z)$ form a basis of $\mathfrak{L}_0$, this implies $x = 0$. This proves (3.9). Hence $j \leq d - k$. If $j = d - k$, the $(d - k) \times (d - k)$ matrix polynomial $W_0(z) := B_{0,b}(z)$ is unimodular. If $j < d - k$ we can extend $B_{0,b}(z)$ to a unimodular $(d - k) \times (d - k)$ matrix polynomial (also denoted by) $W_0(z)$ with $\det W_0(\alpha) \neq 0$ in the same way as the matrix $B(z)$ was extended to $W(z)$ in (3.4) by the means of the Smith normal form. In both cases the first $j$ columns of $W_0(z)$ are the columns of $B_{0,b}(z)$.

Let $W_t(z)$ be the $k \times (d - k)$ matrix obtained from the $k \times j$ matrix $B_{0,t}(z)$ by adding $d - k - j$ zero columns on the right. Define the $d \times d$ matrix polynomial $W(z)$ by

$$W(z) = \begin{bmatrix} I_k & W_t(z) \\ 0_{(d-k)\times k} & W_b(z) \end{bmatrix}.$$ 

Then $W(z)$ is unimodular and $W(z)c_{d,k+l}, l = 1, \ldots, j,$ are the columns of the matrix $B_0(z)$. The operator $W$ acts as the identity on $\mathfrak{C} + S\mathfrak{C}$ and $W\mathcal{C}_m^d = \mathcal{C}_k^d + \mathfrak{L}_0$, where

$$m = k + j = k + n - (\mu_1 + \cdots + \mu_d + k) = n - (\mu_1 + \cdots + \mu_d).$$

Hence $W(\mathcal{C}_m^d + \mathfrak{C} + S\mathfrak{C}) = \mathcal{C}_k^d + \mathfrak{L}_0 + \mathfrak{C} + S\mathfrak{C} = \mathfrak{B}$. \hfill \Box

**Theorem 3.4.** Let $\mathfrak{B}$ be a finite dimensional subspace of $\mathcal{C}_d[z]$. The equality

$$\text{ran}(S_{\mathfrak{B}} - \alpha) = \mathfrak{B} \cap \ker E_\alpha \text{ for all } \alpha \in \mathbb{C} \quad (3.10)$$

holds if and only if there exists a $d \times d$ unimodular matrix polynomial $W(z)$ and a canonical subspace $\mathfrak{C}$ of $\mathcal{C}_d[z]$ such that $\mathfrak{B} = W\mathfrak{C}$.

**Proof.** We first prove the if statement. To prove (3.10) it suffices to show that

$$\mathfrak{B} \cap \ker E_\alpha \subseteq \text{ran}(S_{\mathfrak{B}} - \alpha).$$

Let $f \in \mathfrak{B} \cap \ker E_\alpha$. Then $f(\alpha) = 0$ and $f = W g$ for some $g \in \mathfrak{C}$. Since $W$ is unimodular, $g(\alpha) = 0$. Since $\mathfrak{C}$ is canonical, the polynomial $g(z)/(z - \alpha)$ belongs to $\mathfrak{C}$. Therefore $f(z)/(z - \alpha) = W(z)[g(z)/(z - \alpha)] \in \mathfrak{B}$, hence $f \in \text{ran}(S_{\mathfrak{B}} - \alpha)$.

We prove the only if statement by induction on the dimension of $\mathfrak{B}$. Assume (3.10). The theorem is obviously true if $\dim \mathfrak{B} = 0$. Lemma 3.2 implies that it is true if $\dim \mathfrak{B} = 1$ for then $\mathfrak{B} \cap E_\alpha = \{0\}$. Let $n \in \mathbb{N}$ and state the inductive hypothesis:
If $\mathcal{A}$ is a subspace of $\mathbb{C}^d[z]$ with $\dim \mathcal{A} < n$ and such that
\begin{equation}
\text{ran}(S_\mathcal{A} - \alpha) = \mathcal{A} \cap \ker E_\alpha \quad \text{for all} \quad \alpha \in \mathbb{C},
\end{equation}
then there exists a unimodular $d \times d$ matrix polynomial operator $F(z)$ such that $F\mathcal{A}$ is a canonical subspace of $\mathbb{C}^d[z]$.

Let $\mathcal{B}$ be a finite dimensional subspace of $\mathbb{C}^d[z]$ such that (3.11) holds and $\dim \mathcal{B} = n$. Then $\mathcal{A} = \text{dom } S_\mathcal{B}$ is a proper subspace of $\mathcal{B}$. Therefore $\dim \mathcal{A} < n$. If $\mathcal{A} = \{0\}$, then $\mathcal{B} \cap \ker E_\alpha = \text{ran}(S_\mathcal{B} - \alpha) = \{0\}$ and the theorem follows from Lemma 3.2. Now we assume $\mathcal{A} \neq \{0\}$. By Lemma 3.1 the subspace $\mathcal{A}$ satisfies (3.11).

By the inductive hypothesis there exists a unimodular operator $U$ such that $\ker D = \ker F \mathcal{B}$ and, by (3.10), there exists a $g \in \text{dom } S_\mathcal{B} = \mathcal{A}$ such that $f = S_\mathcal{B}g - \alpha g \in \mathcal{A} + S\mathcal{A}$. Therefore, $Ff \in D + S\mathcal{D}$, which verifies (3.5). Lemma 3.3 applied to $F\mathcal{B}$ yields that there exist a unimodular operator $U$ such that $U^{-1} F\mathcal{B}$ is a canonical subspace of $\mathbb{C}^d[z]$. This proves the theorem with $W = F^{-1}U$.

3.2. The following lemma will be used to deduce Theorem 3.2 from Theorem 3.3.

Lemma 3.5. Let $\mathcal{B}$ be an $n$-dimensional subspace of $\mathbb{C}^d[z]$ and let $B(z)$ be a $d \times n$ matrix polynomial whose columns form a basis of $\mathcal{B}$. Let $l$ be the size of the square diagonal matrix in the Smith normal form (2.2) of $B(z)$. Then
\begin{equation}
\{ \alpha \in \mathbb{C} : \text{ran}(S_\mathcal{B} - \alpha) = \mathcal{B} \cap \ker E_\alpha \} = \{ \alpha \in \mathbb{C} : \text{rank } B(\alpha) = l \}
\end{equation}
if and only if the set on the left hand side is nonempty.

Proof. The if statement follows from the fact that the set on the right hand side in (3.12) is nonempty. Before proving the only if statement we show
\begin{equation}
\dim \text{ran } S_\mathcal{B} \leq \dim \mathcal{B} - l.
\end{equation}

For all $\alpha \in \mathbb{C}$ we have $\text{ran}(S_\mathcal{B} - \alpha) \subseteq \mathcal{B} \cap \ker E_\alpha$, and hence
\begin{equation}
\dim \text{ran } S_\mathcal{B} = \dim \text{ran } (S_\mathcal{B} - \alpha) \leq \dim (\mathcal{B} \cap \ker E_\alpha) = \dim \mathcal{B} - \text{rank } B(\alpha).
\end{equation}

Consequently $l = \max_{\alpha \in \mathbb{C}} \text{rank } B(\alpha) \leq \dim \mathcal{B} - \dim \text{ran } S_\mathcal{B}$. This proves (3.13).

To prove the only if statement assume that $\alpha_0 \in \mathbb{C}$ is in the set on the left hand side of (3.12). Then equality holds in (3.13). Indeed, this follows from
\begin{align*}
\dim \mathcal{B} - l \geq \dim \text{ran } S_\mathcal{B} \\
= \dim \text{ran } (S_\mathcal{B} - \alpha_0) \\
= \dim (\mathcal{B} \cap \ker E_{\alpha_0}) \\
= \dim \mathcal{B} - \text{rank } B(\alpha_0) \\
\geq \dim \mathcal{B} - l.
\end{align*}

Now the equality (3.12) follows from the following sequence of equivalences which hold for all $\alpha \in \mathbb{C}$:
\begin{align*}
\text{rank } B(\alpha) = l & \iff \dim (\mathcal{B} \cap \ker E_{\alpha}) = \dim \mathcal{B} - l \\
& \iff \dim (\mathcal{B} \cap \ker E_{\alpha}) = \dim \text{ ran } S_\mathcal{B} \\
& \iff \dim (\mathcal{B} \cap \ker E_{\alpha}) = \dim \text{ ran } (S_\mathcal{B} - \alpha) \\
& \iff \text{ran}(S_\mathcal{B} - \alpha) = \mathcal{B} \cap \ker E_{\alpha}. \quad \square
Proof of Theorem 1.2 We first prove the if statement. We want to prove that $\mathfrak{B} \cap \ker E_{\alpha} \subseteq \mathrm{ran}(S_{\mathfrak{B}} - \alpha)$. Let $f \in \mathfrak{B} \cap \ker E_{\alpha}$. Then $f(\alpha) = 0$ and $f = Wg$ with $g \in \mathfrak{C}$. Since $W(\alpha)$ is invertible, $g(\alpha) = 0$. As $\mathfrak{C}$ is canonical, the polynomial $h(z) = g(z)/(z - \alpha)$ belongs to $\mathfrak{C}$. Therefore $W(z)h(z) \in \mathfrak{B}$ and $(x - \alpha)W(z)h(z) = f(z)$, which implies $f \in \mathrm{ran}(S_{\mathfrak{B}} - \alpha)$.

To prove the only if statement, assume that (1.2) holds for $\alpha = \alpha_0$. Let $B(z)$ be a $d \times n$ matrix polynomial whose columns form a basis of $\mathfrak{B}$. Let

$$B(z) = U(z) \begin{bmatrix} D(z) & 0 \\ 0 & 0 \end{bmatrix} V(z)$$

be the Smith normal form (2.2) of $B(z)$ where $D(z)$ is an $l \times l$ diagonal matrix with nonzero diagonal entries. Now define the space $\mathfrak{B}_1 \subseteq \mathbb{C}^d[z]$ as the span over $\mathbb{C}$ of the columns of

$$B_1(z) = U(z) \begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix} V(z).$$

Set

$$F(z) = U(z) \begin{bmatrix} D(z) & 0 \\ 0 & I_{d-l} \end{bmatrix} U(z)^{-1}.$$

Then $\mathfrak{B} = F\mathfrak{B}_1$ and $\det F(\alpha_0) \neq 0$. Moreover, since

$$\{ \alpha \in \mathbb{C} : \det F(\alpha) \neq 0 \} = \{ \alpha \in \mathbb{C} : \mathrm{rank} B(\alpha) = l \}$$

and by Lemma 3.3, (1.3) holds for $F(z)$. From $\det F(\alpha_0) \neq 0$ it follows that

$$\mathrm{ran}(S_{\mathfrak{B}} - \alpha_0) = \mathfrak{B} \cap \ker E_{\alpha_0} \quad \Rightarrow \quad \mathrm{ran}(S_{\mathfrak{B}_1} - \alpha_0) = \mathfrak{B}_1 \cap \ker E_{\alpha_0}.$$

Since $\mathrm{rank} B_1(\alpha) = l$ for all $\alpha \in \mathbb{C}$, Lemma 3.5 implies that

$$\mathrm{ran}(S_{\mathfrak{B}_1} - \alpha) = \mathfrak{B}_1 \cap \ker E_{\alpha} \quad \text{for all} \quad \alpha \in \mathbb{C}.$$

By Theorem 3.6 there exists a unimodular matrix $U(z)$ such that $\mathfrak{C} = U^{-1} \mathfrak{B}$ is a canonical subspace of $\mathbb{C}^d[z]$, hence $\mathfrak{B} = W\mathfrak{C}$ with $W = FU$. Finally, (1.3) holds, because $U$ is unimodular and $F(z)$ satisfies (1.3).  

3.3. Theorem 1.2 can also be formulated in terms of matrix polynomials:

**Theorem 3.6.** Let $\mathfrak{B}$ be an $n$-dimensional subspace of $\mathbb{C}^d[z]$, $n \geq 1$. Let $B(z)$ be a $d \times n$ matrix polynomial whose columns form a basis of $\mathfrak{B}$. Let $b_1(z)$ and $l$ be as in the Smith normal form (2.2) of $B(z)$. Then $l + \mathrm{dim} \, S_{\mathfrak{B}} = \dim \mathfrak{B}$ if and only if there exist

(a) a $d \times d$ matrix polynomial $W(z)$ whose determinant has the same zeros as $b_1(z)$,
(b) nonnegative integers $\delta_0 \geq \delta_1 \geq \cdots \geq \delta_m$ with $\delta_0 + \cdots + \delta_m = n$ and
(c) an invertible $n \times n$ constant matrix $T$

such that

$$B(z) = W(z) \begin{bmatrix} P_{\delta_0} & P_{\delta_1} z & \cdots & P_{\delta_m} z^m \end{bmatrix} T \quad \text{for all} \quad z \in \mathbb{C},$$

where $P_{\delta}$ stands for the $d \times \delta$ matrix: $P_{\delta} = \begin{bmatrix} I_{\delta} & 0 \end{bmatrix}^\top$. 


Proof. For all $\alpha \in \mathbb{C}$ we have $\ker (S_\mathcal{B} - \alpha) \subseteq \mathcal{B} \cap \ker E_\alpha$. For all $\alpha \in \mathbb{C}$ with $b_1(\alpha) \neq 0$ we have

$$l + \dim \text{dom} S_\mathcal{B} = \dim \text{ran} (E_\alpha | \mathcal{B}) + \dim \text{ran} (S_\mathcal{B} - \alpha)$$

$$\leq \dim \text{ran} (E_\alpha | \mathcal{B}) + \dim (\mathcal{B} \cap \ker E_\alpha)$$

$$= \dim \mathcal{B}$$

and equality holds if and only if $\ker (S_\mathcal{B} - \alpha) = \mathcal{B} \cap \ker E_\alpha$.

To prove the only if statement, assume $l + \dim \text{dom} S_\mathcal{B} = \dim \mathcal{B}$. Then we can apply Theorem 1.2: There exist a matrix polynomial $W(z)$ satisfying (a) and a canonical subspace $\mathcal{C}$ of $C^d[z]$ such that $\mathcal{B} = W \mathcal{C}$. Let $\mu_1 \geq \cdots \geq \mu_d$ be the degrees of $\mathcal{C}$. Since $n \geq 1$, we have $\mu_1 \geq 1$. Set $m = \mu_1 - 1$ and $\delta_j = \# \{ i \in \{1, \ldots, d\} : \mu_i > j \}, \quad j \in \{0, \ldots, m\}$.

Then the equality in (b) holds. Since the columns of the matrix $[P_{ka} \cdots P_{ka} z^m]$ form a basis for $\mathcal{C}$, there exists a matrix $T$ satisfying (c) such that (3.14) holds.

To prove the if statement, we note that (a)-(c) and (3.14) imply that $\mathcal{B} = W \mathcal{C}$ with $\mathcal{C}$ as above, and hence Theorem 1.2 can be applied and together with the if and only if statement at the beginning of the proof yield that $l + \dim \text{dom} S_\mathcal{B} = \dim \mathcal{B}$. □

Remark 3.7. Denote by $(S^j)_{\mathcal{B}}$ the range restriction of $S^j$ to $\mathcal{B}$. Then in item (b) of Theorem 3.6: $m$ is the nonnegative integer with

$\{0\} = \text{dom} (S^{m+1})_{\mathcal{B}} \subsetneq \text{dom} (S^m)_{\mathcal{B}}$

and

$$\delta_j = \dim \text{dom}(S^j)_{\mathcal{B}} - \dim \text{dom}(S^{j+1})_{\mathcal{B}} \quad j \in \{0, \ldots, m\}.$$ 

Moreover, if we set $\delta_{-1} = d$ then the numbers

$$\mu_k = 1 + \max \{ j \in \{-1, 0, \ldots, m\} : \delta_j \geq k \}, \quad k \in \{1, \ldots, d\},$$

are the Forney indices of the matrix polynomial $B(\alpha)$ and the degrees of the canonical space $W^{-1} \mathcal{B}$.

4. Proof of Theorem 1.1

4.1. We divide the proof of Theorem 1.1 in two parts.

Proof of the if statements in Theorem 1.1. Assume (A) and (B). We show that $\mathcal{B}$ has a reproducing Nevanlinna kernel in steps (i)-(iv). In step (v) we prove the last if statement in the theorem.

(i) By Theorem 1.2 there exist a canonical subspace $\mathcal{C} \subseteq C^d[z]$ with degrees $\mu_1 \geq \cdots \geq \mu_d \geq 0$ and a $d \times d$ matrix polynomial $W(z)$ with $\det W(\alpha) \neq 0$ such that $\mathcal{B} = W \mathcal{C}$. Set $n = \dim \mathcal{B}$ and let $l \in \{1, \ldots, \min \{d, n\}\}$ be the integer introduced in Remark 2.2. Then the defect index of the symmetric operator $S_\mathcal{B}$ is $(l, l)$, or equivalently,

$$\mu_1 \geq \cdots \geq \mu_l \geq 1 \quad \text{and} \quad \mu_{l+1} = \cdots = \mu_d = 0.$$ 

It follows that the elements of $\mathcal{B}$ are of the form:

$$f(z) \in \mathcal{B} \Rightarrow f(z) = W(z) \begin{bmatrix} x(z) \\ 0 \end{bmatrix},$$
implies that for each \( w \in \mathbb{C} \). Let \( w \in \mathcal{K} \) as \( K(z, w) = B(z)G^{-1}B(w)^* \), \( z, w \in \mathbb{C} \). By \([13]\), this representation implies that for each \( w \in \mathbb{C} \) which belongs to the set in \((3.12)\), the columns of \( K(z, w) \) span an \( l \)-dimensional subspace of \( \mathfrak{B} \), in formula:

\[
(4.2) \quad \dim \{K(\cdot, w)x : x \in \mathbb{C}^d\} = l \quad \text{whenever} \quad w \in \{\alpha \in \mathbb{C} : \text{rank } B(\alpha) = l \}.
\]

(ii) In the following we use graph notation in the space \( \mathfrak{B} \oplus \mathfrak{B} \). The operator \( S_{2\mathfrak{B}}^* \) is identified with its graph in \( \mathfrak{B} \oplus \mathfrak{B} \) and its adjoint \( S_{2\mathfrak{B}} \) is the orthogonal complement of \( S_{2\mathfrak{B}}^* \) in \( \mathfrak{B} \oplus \mathfrak{B} \) equipped with the Lagrange inner product

\[
[(f, g), (p, q)] = -i([g, p]_{\mathfrak{B}} - [f, q]_{\mathfrak{B}}), \quad (f, g), (p, q) \in \mathfrak{B} \oplus \mathfrak{B}.
\]

Let \( w \in \mathbb{C}, x \in \mathbb{C}^d \) and \( (f, Sf) \in S_{2\mathfrak{B}} \) be arbitrary. Then

\[
[(f, Sf), \{K(\cdot, w)x, w^*K(\cdot, w)x\}] = -i([Sf, K(\cdot, w)x]_{\mathfrak{B}} - [f, w^*K(\cdot, w)x]_{\mathfrak{B}})
\]

\[
= -i(x^*(Sf)(w) - wx^*f(w)) = 0
\]

and hence \( \{K(\cdot, w)x, w^*K(\cdot, w)x\} : x \in \mathbb{C}^d \} \subseteq S_{2\mathfrak{B}}^* \cap (w^*I) \) for all \( w \in \mathbb{C} \). According to the definition of defect number \((\text{see } [9, \text{p.369}])\) and by \((4.2)\), it follows that for all \( w \in \{\alpha \in \mathbb{C} \setminus \mathbb{R} : \text{rank } B(\alpha) = l \} \)

\[
(4.3) \quad \{K(\cdot, w)x, w^*K(\cdot, w)x\} : x \in \mathbb{C}^d \} = S_{2\mathfrak{B}}^* \cap (w^*I),
\]

because for such \( w \)'s both sets have dimension \( l \). Consider the subspace

\[
(4.4) \quad \mathfrak{L}_0 := \text{span} \{\{K(\cdot, w)x, w^*K(\cdot, w)x\} : w \in \mathbb{C}, x \in \mathbb{C}^d \}
\]

of \( S_{2\mathfrak{B}}^* \). Since \( S_{2\mathfrak{B}} \) has no eigenvalues, the generalized von Neumann formula given in \([9]\) Theorem 3.7] implies that for \( d + 1 \) distinct points \( w_0, \ldots, w_d \) from the set \( \{\alpha \in \mathbb{C} : \text{Im } \alpha > 0, \text{rank } B(\alpha) = l \} \) we have

\[
S_{2\mathfrak{B}}^* = S_{2\mathfrak{B}} + S_{2\mathfrak{B}}^* \cap (w_0^*I) + \sum_{j=1}^d S_{2\mathfrak{B}}^* \cap (w_j^*I).
\]

Combined with \([13]\) and \((4.4)\) this yields \( S_{2\mathfrak{B}} + \mathfrak{L}_0 \subseteq S_{2\mathfrak{B}}^* \subseteq S_{2\mathfrak{B}} + \mathfrak{L}_0 \), and hence

\[
(4.5) \quad S_{2\mathfrak{B}}^* = S_{2\mathfrak{B}} + \mathfrak{L}_0.
\]

(iii) Let \( (\mathfrak{B}_1, [\cdot, \cdot]_{\mathfrak{B}_1}) \) be the reproducing kernel Pontryagin space whose kernel is

\[
L_1(z, w) = i(z - w^*)K(z, w), \quad z, w \in \mathbb{C}.
\]

We claim that its positive and negative index are \( l \). To prove the claim we consider the operator \( T : (S_{2\mathfrak{B}}^*, [\cdot, \cdot]) \to \mathfrak{B}_1 \) defined by \( T([f, g]) = Sf - g, \ (f, g) \in S_{2\mathfrak{B}}^* \), and show that it is a partial isometry onto \( \mathfrak{B}_1 \) with null space \( \ker T = S_{2\mathfrak{B}} \). The last equality is easy to verify. That \( \text{ran } T = \mathfrak{B}_1 \) follows from \((4.3)\) as it implies (with \( w \in \mathbb{C} \) and \( x \in \mathbb{C}^d \)):

\[
(T(\{K(\cdot, w)x, w^*K(\cdot, w)x\})(z) = (z - w^*)K(z, w)x = -iL_1(z, w)x.
\]

That \( T \) is isometric follows from \((4.5)\), the symmetry of \( S_{2\mathfrak{B}} \) and the equalities (with \( w, v \in \mathbb{C} \) and \( x, y \in \mathbb{C}^d \)):

\[
\left( \{K(\cdot, w)x, w^*K(\cdot, w)x\}, \{K(\cdot, v)y, v^*K(\cdot, v)y\} \right)
\]
exists an \( l \) \( \in (B) \) belongs to the sets in (3.12) and (1.3) and hence \( d \) where now \( 0 \) stands for the \( \langle Q \rangle \).

Then (iv) Let \( B_1(z) \) be a \( d \times 2l \) matrix polynomial whose columns form a basis of \( \mathfrak{B}_1 \), and let \( Q_1 \) be the corresponding \( 2l \times 2l \) Gram matrix. Then \( Q_1 \) is self-adjoint and, by the claim proved in (iii), has \( l \) positive and \( l \) negative eigenvalues. Let \( B_2(z) \) be the \( d \times 2(d-l) \) matrix polynomial defined by

\[
B_2(z) = W(z) \begin{bmatrix} 0 & 0 \\ I_{d-l} & I_{d-l} \end{bmatrix},
\]

where the zero matrices are of size \( l \times (d-l) \). Define the \( d \times 2d \) matrix polynomial \( P(z) \) by \( P(z) = [B_1(z) \ B_2(z)] \) and the \( 2d \times 2d \) block diagonal matrix \( Q \) by

\[
Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad \text{where} \quad Q_2 = \begin{bmatrix} 0 & iI_{d-l} \\ -iI_{d-l} & 0 \end{bmatrix}.
\]

Then \( Q \) is self-adjoint and has \( d \) positive and \( d \) negative eigenvalues. We claim that

(I) \( \text{rank} P(z) = d \) for some \( z \in \mathbb{C} \), and

(II) \( P(z)Q P(z)^* = i(z - w^*)K(z, w) \) for all \( z, w \in \mathbb{C} \).

We prove (II): The inclusion \( \mathfrak{B}_1 = T(S_{\mathbb{B}}^*) \subseteq \mathfrak{B} + S \mathfrak{B} \) and (1.1) imply that there exists an \( l \times 2l \) matrix polynomial \( X(z) \) such that

\[
B_1(z) = W(z) \begin{bmatrix} X(z) & 0 \\ 0 & 0 \end{bmatrix},
\]

where now \( 0 \) stands for the \( (d-l) \times 2l \) zero matrix. The complex number \( \alpha \) satisfying \( (B) \) belongs to the sets in (3.12) and (1.3) and hence

\[
\text{rank} X(\alpha) = \text{rank} B_1(\alpha)
\]

\[
= \text{dim} \ E_\alpha \mathfrak{B}_1
\]

\[
= \text{dim} \ \text{span} \ \{ L_1(\alpha, w)x : w \in \mathbb{C}, x \in \mathbb{C}^d \}
\]

\[
= \text{dim} \ \text{span} \ \{ i(\alpha - w^*)K(\alpha, w)x : w \in \mathbb{C}, x \in \mathbb{C}^d \}
\]

\[
= \text{dim} \ \text{span} \ \{ K(\alpha, w)y : w \in \mathbb{C}, y \in \mathbb{C}^d \}
\]

\[
= \text{dim} \ E_\alpha \mathfrak{B}
\]

\[
= \text{rank} B(\alpha)
\]

\[
= l.
\]

The equality

\[
P(z) = W(z) \begin{bmatrix} X(z) & 0 & 0 \\ 0 & I_{d-l} & I_{d-l} \end{bmatrix}
\]
implies that \( \text{rank} \mathbf{P}(\alpha) = d \). This proves (I). We prove (II):

\[ \mathbf{P}(z)Q^{-1}\mathbf{P}(w)^* = B_1(z)Q_1^{-1}B_1(w)^* + B_2(z)Q_2^{-1}B_2(w)^* \]

\[ = L_1(z, w) + W(z) \begin{bmatrix} 0 & 0 \\ I_{d-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & iI_{d-1} \\ 0 & 0 \end{bmatrix} W(w)^* \]

\[ = i(z - w^*)K(z, w). \]

Items (I) and (II) show that \( K(z, w) \) is a polynomial Nevanlinna kernel for \( \mathfrak{B} \). This completes the proof of the if statement.

(v) If (B) holds for all \( \alpha \in \mathbb{C} \), then, by Theorem 3.4, \( W(z) \) is unimodular and the proof of (I) shows that \( \text{rank} \mathbf{P}(z) = d \) for all \( z \in \mathbb{C} \). \( \square \)

4.2. In the proof of the only if statements in Theorem 1.1 we use the following lemma.

**Lemma 4.1.** Let \( Q \) be a self-adjoint \( 2d \times 2d \) matrix with \( d \) positive and \( d \) negative eigenvalues. Let \( \mathbf{P}(z) \) be a \( d \times 2d \) matrix polynomial such that

(a) \( \mathbf{P}(z)Q^{-1}\mathbf{P}(z)^* = 0 \) for all \( z \in \mathbb{C} \),
(b) \( \text{rank} \mathbf{P}(z) = d \) for all \( z \in \mathbb{C} \), and
(c) \( \mathbf{P}(z) \) is row reduced and has row degrees \( \sigma_1, \ldots, \sigma_d \), assumed ordered so that \( \sigma_1 \geq \cdots \geq \sigma_d \) and \( \sigma_1 = \deg \mathbf{P}(z) =: p \).

Equip \( \mathbb{C}^{2d}[z]_{<p} \) with the inner product

\[ [f, g]_Q = \sum_{j=0}^{p-1} b_{p-1-j} Q^{-1} a_j, \quad f(z) = \sum_{j=0}^{p-1} a_j z^j, \quad g(z) = \sum_{j=0}^{p-1} b_j z^j, \quad a_j, b_j \in \mathbb{C}^d, \]

and consider the following subspace of \( \mathbb{C}^{2d}[z]_{<p} \):

\[ \mathcal{L}_p = \text{span} \left\{ \sum_{k=0}^{p-1} z^{p-1-k} w^{*k} \mathbf{P}(w)^* x : w \in \mathbb{C}, \ x \in \mathbb{C}^d \right\}. \]

Then the orthogonal complement of \( \mathcal{L}_p \) in \( (\mathbb{C}^{2d}[z]_{<p}, [\cdot, \cdot]_Q) \) is

\[ \mathcal{L}_p^\perp = \left\{ f(z) \in \mathbb{C}^{2d}[z]_{<p} : f(z) = \mathbf{P}(z)^* u(z) \text{ with } u(z) \in \mathbb{C}^d[z] \right\}. \] (4.6)

It is the isotropic part of \( \mathcal{L}_p \) and \( \mathcal{L}_p/\mathcal{L}_p^\perp \) is a Pontryagin space with positive and negative index \( \sigma_1 + \cdots + \sigma_d \).

**Proof.** For an element \( f(z) = \sum_{j=0}^{p-1} a_j z^j \in \mathbb{C}^{2d}[z]_{<p} \) the following equivalences hold:

\[ f(z) \in \mathcal{L}_p^\perp \iff \left( \sum_{k=0}^{p-1} w^{*k} a_k^* \right) Q^{-1} \mathbf{P}(w)^* = 0 \quad \text{for all } w \in \mathbb{C}, \]

\[ \iff \mathbf{P}(z)Q^{-1} f(z) = 0 \quad \text{for all } z \in \mathbb{C}, \]

\[ \iff f(z) = \mathbf{P}(z)^* u_z \quad \text{for some } u_z \in \mathbb{C}^d \text{ and all } z \in \mathbb{C}. \]

The last equivalence follows from (22) and (14). To prove that the vector \( u_z \) depends polynomially on \( z \) we use that the Smith normal form (22.2) of \( \mathbf{P}(z) \) is given by:
The number on the right hand side can be expressed in terms of the Forney indices $P$.

Therefore, dim $H$ whose dimension is $dp$.

To prove the last two statements of the lemma we apply Lemma 2.5 with $n = dp$ and $\tau = dp - (\sigma_1 + \cdots + \sigma_d)$. The assumptions about $Q$ in the lemma readily imply that $\mathbb{C}^{2dp}[z]_{<p}$ is a $2dp$-dimensional Pontryagin space with negative index $dp$. It remains to construct a maximal neutral subspace of $\mathbb{C}^{2dp}[z]_{<p}$ which is contained in $\mathcal{L}_p$. We begin with the subspace $\mathcal{H} = \text{ran } H$, where the operator $H : \mathbb{C}^{d}[z] \to \mathbb{C}^{2d}[z]$ maps $u(z) \in \mathbb{C}^{d}[z]$ into the polynomial part of $\mathbf{P}(1/z^*) u(z)$. For example, if $\mathbf{P}(z)$ is written as:

$$\mathbf{P}(z) = \mathbf{P}(0) + z\mathbf{P}_1 + \cdots + z^p\mathbf{P}_p,$$

then for $k \in \{0\} \cup \mathbb{N}$ and $x \in \mathbb{C}^d$

$$H(z^k x) = \begin{cases} 
\sum_{j=0}^{k} z^j \mathbf{P}_{k-j} x & \text{if } k < p, \\
\sum_{j=0}^{p} z^j \mathbf{P}_{p-j} x & \text{if } p \leq k.
\end{cases}$$

These formulas imply that $\mathcal{H}$ is neutral in $\mathbb{C}^{2d}[z]_{<p}$. Indeed, for $k, m \in \{0\} \cup \mathbb{N}$ and $x, y \in \mathbb{C}^d$ we have

$$\left[H(z^k x), H(z^m y)\right]_Q = \sum_{j=0}^{\min\{k, m\}} y^* \mathbf{P}_{p-1-m+j} \mathbf{Q}^{-1} \mathbf{P}_{k-j}^* x = \sum_{i+j = p-1-m+k, i, j \in \{0,...,p\}} y^* \mathbf{P}_i \mathbf{Q}^{-1} \mathbf{P}_j^* x$$

and the last expression equals 0 because the assumption (18) is equivalent to

$$\sum_{j+k=n, j, k \in \{0,...,p\}} \mathbf{P}_j \mathbf{Q}^{-1} \mathbf{P}_k = 0 \quad \text{for all } n \in \{0,...,2p\}.$$

Since, by (16), $\mathbf{P}_0 = \mathbf{P}(0)$ has full rank, $H$ is degree preserving and hence injective. Therefore, dim $\mathcal{H} = dp = (1/2) \dim(\mathbb{C}^{2d}[z]_{<p})$ and $\mathcal{H}$ is maximal neutral.
Define the mapping \( R : \mathbb{C}^{2d}[z]_{<p} \to \mathbb{C}^{2d}[z]_{<p} \) by \((Rf)(z) = z^{p-1}f(1/z)\). Then \( R \) is unitary with respect to \([\cdot, \cdot]_Q\) and hence \( \mathcal{H} := R\mathcal{H} \) is also a maximal neutral subspace of \( \left( \mathbb{C}^{2d}[z]_{<p}, [\cdot, \cdot]_Q \right) \). The proof of the lemma is complete if we show that \( \mathcal{H} \subseteq \mathcal{L}_p \). For that we consider the polynomials of the form

\[
(4.8) \quad \sum_{k=0}^{2p-1} z^k w^* k x, \quad w \in \mathbb{C}, \quad x \in \mathbb{C}^d.
\]

From

\[
P\left( \frac{1}{z^*} \right) \left( \sum_{k=0}^{2p-1} z^k w^* k x \right) = \sum_{j=0}^{2p-1} \left( \frac{1}{z^*} \sum_{k=0}^{2p-1} z^k w^* k \right) P_j^* x
\]

and

\[
\frac{1}{z^*} \sum_{k=0}^{2p-1} z^k w^* k = \sum_{j=0}^{2p-1} w^* x + w^* \sum_{k=0}^{2p-1} z^k w^* k + \sum_{k=p}^{2p-1} z^k w^*(k+j)
\]

we obtain

\[
(4.9) \quad H \left( \sum_{k=0}^{2p-1} z^k w^* k x \right) = \sum_{j=0}^{2p-1} z^k w^* k P(w)^* x + \text{higher order terms.}
\]

Since the space \( \mathbb{C}^d[z]_{<2p} \) is spanned by polynomials in \( \left(1, z, z^2, \ldots, z^{2p-1}\right) \), each element of \( \mathbb{C}^d[z]_{<2p} \) is also a sum of polynomials in \( \left(1, z, z^2, \ldots, z^{2p-1}\right) \). As \( H \) is degree preserving, the polynomials in \( \mathcal{H} = H\left( \mathbb{C}^d[z]_{<2p} \right) \) have degrees \(<p \) and therefore they have the form \( \left(4.9\right) \) with zero higher order terms. Thus

\[
\mathcal{H} \subset \text{span} \left\{ \sum_{k=0}^{2p-1} z^k w^* k P(w)^* x : w \in \mathbb{C}, x \in \mathbb{C}^d \right\}
\]

and \( \mathcal{H} = R\mathcal{H} \subseteq \mathcal{L}_p \). This proves Lemma 4.1. \( \square \)

**Proof of the only if statements in Theorem 1.1** Assume that the reproducing kernel of \( \mathcal{B} \) is a polynomial Nevanlinna kernel \( K(z, w) \):

\[
i (z - w^*) K(z, w) = P(z) Q^{-1} P(w)^* \quad \text{for all} \quad z, w \in \mathbb{C},
\]

where \( Q \) is a self-adjoint \( 2d \times 2d \) matrix with \( d \) positive and \( d \) negative eigenvalues and \( P(z) \) is a \( d \times 2d \) matrix polynomial with rank \( P(z) = d \) for some \( z \in \mathbb{C} \). Note that \( \left(4.10\right) \) implies \( \left(4.11\right) \) of Lemma 4.1.

\[
P(z) Q^{-1} P(z)^* = 0 \quad \text{for all} \quad z \in \mathbb{C}.
\]

We prove \( \left(\text{A}\right) \) and \( \left(\text{B}\right) \) in the steps (i)–(iv), in step (v) we prove the last only if statement in the theorem.

(i) In this step we prove \( \left(\text{B}\right) \) under the assumption that \( \left(\text{A}\right) \) and \( \left(\text{B}\right) \) of Lemma 4.1 hold. Denote by \( \mathcal{B}_p \) the reproducing kernel Pontryagin space with kernel

\[
L_p(z, w) = i (z^p - w^p) K(z, w) = \left( \sum_{k=0}^{p-1} z^{p-1-k} w^* k \right) P(z) Q^{-1} P(w)^*, \quad z, w \in \mathbb{C}.
\]

Then

\[
\mathcal{B}_p = \text{span} \left\{ P(z) Q^{-1} \sum_{k=0}^{p-1} z^{p-1-k} w^* k P(w)^* x : w \in \mathbb{C}, x \in \mathbb{C}^d \right\}
\]
This completes the proof of (4.14).

Comparing this inner product with the one defined in Lemma 4.1, we find that \( P(z)Q^{-1} \) considered as a multiplication operator maps \( \mathfrak{L}_p \subset \mathbb{C}^{2d}[z]_{<p} \) isometrically onto \( \mathfrak{B}_p \) and its null space is \( \mathfrak{L}^{\perp}_p \) (see the second of the three equivalences in the beginning of the proof of Lemma 4.1). Hence, \( \dim \mathfrak{B}_p = 2(\sigma_1 + \cdots + \sigma_d) \) and the positive and the negative index of \( \mathfrak{B}_p \) equal \( \sigma_1 + \cdots + \sigma_d \). According to Lemma 2.7 we have \( \dim \mathfrak{B} = \sigma_1 + \cdots + \sigma_d \). The space \( \mathfrak{B} \) is spanned by the columns of \( K(z, w) \), \( w \in \mathbb{C} \) and for \( j \in \{1, \ldots, d\} \) the degree of the \( j \)-th row of \( K(z, w) \) as a polynomial in \( z \) is equal to \( \max\{0, \sigma_j - 1\} \). Therefore \( \mathfrak{B} \subset \bigoplus_{j=1}^d (\mathbb{C}[z]_{<\sigma_j})^{e_{d,k}} \). Since both spaces have dimension \( \sigma_1 + \cdots + \sigma_d \), equality prevails:

\[(4.12) \quad \mathfrak{B}_0 = \bigoplus_{j=1}^d (\mathbb{C}[z]_{<\sigma_j})^{e_{d,k}}.\]

This implies (ii).

(ii) In this step we prove (A) under the assumption that (i) and (ii) of Lemma 4.1 hold. Set

\[
M(z) = \begin{bmatrix}
z^{-\sigma_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & z^{-\sigma_d}
\end{bmatrix}.
\]

Then \( P_\infty = \lim_{z \to \infty} M(z)P(z) \) and by (4.11) we have

\[(4.13) \quad P_\infty Q^{-1}P_\infty^* = \lim_{z \to \infty} M(z)P(z)Q^{-1}P(z)^*D(z)^* = 0.\]

Since \( P_\infty \) has full rank, (4.13) implies that the linear span of the columns of \( P_\infty^* \) is a maximal neutral subspace of \( (\mathbb{C}^{2d}, [\cdot, \cdot]_Q) \) and this span coincides with the null space of \( P_\infty Q^{-1} \). We claim that for \( a \in \mathbb{C}^{2d} \)

\[(4.14) \quad P(z)a \in \mathfrak{B} \quad \iff \quad P_\infty a = 0.\]

To prove the claim assume first that \( P(z)a \in \mathfrak{B} \). From (4.12) we see that the degree of the \( j \)-th entry of the vector polynomial \( P(z)a \) is strictly less then \( \sigma_j, j \in \{1, \ldots, d\} \). Hence \( P_\infty a = \lim_{z \to \infty} M(z)(P(z)a) = 0 \). As to the converse, first notice that by the definition of \( P_\infty \) the row degrees of the matrix polynomial \( P_0(z) = P(z) - M(z)^{-1}P_\infty \) are strictly less than \( \sigma_j, j \in \{1, \ldots, d\} \). By (4.12) we have that \( P_0(z)a \in \mathfrak{B} \) for all \( a \in \mathbb{C}^{2d} \). Now assume \( P_\infty a = 0 \). Then

\[P(z)a = P_0(z)a + M(z)^{-1}P_\infty a = P_0(z)a \in \mathfrak{B}.\]

This completes the proof of (4.14).

Consider \( f \in \mathfrak{B} \). Since \( \mathfrak{B} \) is finite dimensional it can be written as

\[(4.15) \quad f(z) = \sum_{i=1}^m K(z, w_i)x_i, \quad m \in \mathbb{N}, \ w_i \in \mathbb{C}, \ x_i \in \mathbb{C}^d, \ i \in \{1, \ldots, m\}.\]
The next sequence of equivalences follows from (4.14) and the observation after (4.13):

\[
\begin{align*}
\text{if } f \in \text{dom } S_B & \iff Sf \in \mathcal{B} \\
& \iff \sum_{i=1}^{m} (z - w_i^*) K(z, w_i) x_i \in \mathcal{B} \\
& \iff P(z) Q^{-1} \left( \sum_{i=1}^{m} P(w_i)^* x_i \right) \in \mathcal{B} \\
& \iff P_\infty Q^{-1} \left( \sum_{i=1}^{m} P(w_i)^* x_i \right) = 0 \\
& \iff \sum_{i=1}^{m} P(w_i)^* x_i = P_\infty^* x \quad \text{for some } x \in \mathbb{C}^d.
\end{align*}
\]

Let \( f \in \mathcal{B} \) be given by (4.15) and let \( g \in \mathcal{B} \) be of the form

\[
g(z) = \sum_{j=1}^{n} K(z, v_j) y_j, \quad n \in \mathbb{N}, \ v_j \in \mathbb{C}, \ y_j \in \mathbb{C}^d, \ j \in \{1, \ldots, n\}.
\]

Assume that \( f, g \in \text{dom } S_B \). Then there exist \( x, y \in \mathbb{C}^d \) such that

\[
\sum_{i=1}^{m} P(w_i)^* x_i = P_\infty^* x \quad \text{and} \quad \sum_{i=1}^{m} P(v_i)^* y_i = P_\infty^* y
\]

and using the reproducing kernel property of \( K(z, w) \) we have

\[
[Sf, g]_B - [f, Sg]_B = \sum_{j=1}^{n} \sum_{i=1}^{m} v_j y_j^* K(v_j, w_i) x_i - \left( \sum_{i=1}^{m} \sum_{j=1}^{n} w_i x_i^* K(w_i, v_j) y_j \right)^* \\
= \sum_{j=1}^{n} \sum_{i=1}^{m} v_j y_j^* K(v_j, w_i) x_i - \sum_{i=1}^{m} \sum_{j=1}^{n} w_i x_i^* K(v_j, w_i) x_i \\
= \sum_{j=1}^{n} \sum_{i=1}^{m} (v_j - w_i^*) y_j^* K(v_j, w_i) x_i \\
= -i \sum_{j=1}^{n} \sum_{i=1}^{m} y_j^* P(v_j) Q^{-1} P(w_i)^* x_i \\
= -i \left( \sum_{j=1}^{n} y_j^* P(v_j) \right) Q^{-1} \left( \sum_{i=1}^{m} P(w_i)^* x_i \right) \\
= -i y^* P_\infty Q^{-1} P_\infty^* x \\
= 0.
\]

This proves that \( S_B \) is symmetric.

(iii) In this step we only assume \( (b) \) of Lemma 4.1: \( \text{rank } P(z) = d \) for all \( z \in \mathbb{C} \). Then there is a unimodular \( d \times d \) matrix polynomial \( U(z) \) such that \( S(z) = U(z) P(z) \) is row reduced with ordered row degrees \( \sigma_1 \geq \cdots \geq \sigma_d \). Then \( U \) is an isometry.
from \( \mathcal{B} \) onto the reproducing kernel Pontryagin space \( \mathcal{C} \) with kernel

\[
-i \frac{S(z)Q^{-1}S(w)^*}{z - w^*}.
\]

According to what has already been proved in (i)

\[
U \mathcal{B} = \mathcal{C} = \bigoplus_{j=1}^{d} (\mathbb{C}[z]_{<\sigma_j}) e_{d,k}.
\]

Thus \( \mathcal{B} = U^{-1} \mathcal{C} \) and, by Theorem 3.4, (B) holds for all \( \alpha \in \mathbb{C} \). According to part (ii) of this proof, \( S_{U\mathcal{B}} \) is symmetric, hence \( S_{\mathcal{B}} = U^{-1} S_{U\mathcal{B}} U \) is also symmetric, that is, (A) holds.

(iv) Finally we prove that (A) and (B) hold if \( \text{rank } P(z) = d \) for some \( z \in \mathbb{C} \) as in the beginning of this proof. In that case there exist a \( d \times d \) matrix polynomial \( G(z) \) with \( \det G(z) \neq 0 \) and a \( d \times 2d \) matrix polynomial \( S(z) \) with \( \text{rank } S(z) = d \) for all \( z \in \mathbb{C} \) such that \( P(z) = G(z)S(z) \) for all \( z \in \mathbb{C} \), see Lemma 2.4. If by \( \mathfrak{A} \) we denote the reproducing kernel space with Nevanlinna kernel

\[
-i \frac{S(z)QS(w)^*}{z - w^*},
\]

then, by what has been proved in (iii), the operator \( S_{\mathfrak{A}} \) is symmetric and for almost all \( \alpha \in \mathbb{C} \) we have \( \text{ran}(S_{\mathfrak{A}} - \alpha) = \mathfrak{A} \cap \ker E_\alpha \). Now (A) and (B) follow since the multiplication operator \( G \) corresponding to \( G(z) \) is an isomorphism from \( \mathfrak{A} \) onto \( \mathcal{B} \).

(v) The last only if statement in the theorem follows from step (iii) above and Theorem 3.4. \( \square \)

5. \( Q \)-functions

5.1. Let \( M(z) \) be a generalized Nevanlinna \( d \times d \) matrix function and denote by \( \mathcal{L}(M) \) the reproducing kernel Pontryagin space with reproducing kernel \( K_M(z, w) = K_{M,I_d}(z, w) \). By [16, Theorem 2.1], the operator \( S \) in \( \mathcal{L}(M) \) of multiplication by the independent variable is a simple symmetric operator with equal defect numbers and its adjoint is given by

\[
S^* = \text{span} \left\{ \{K_M(\cdot, w^*)x, wK_M(\cdot, w^*)x \} : x \in \mathbb{C}^d, w \in \text{hol } (M) \right\}
\]

\[
= \left\{ \{f, g\} \in \mathcal{L}(M)^2 : \exists x, y \in \mathbb{C}^d \text{ such that } g(z) - zf(z) \equiv x - M(z)y \right\}.
\]

If follows that for all \( w \in \text{hol } (M) \)

\[
\ker(S^* - w) = \{K_M(\cdot, w^*)x : x \in \mathbb{C}^d\} = \text{ran } E_w^*,
\]

where \( E_w \) is considered as a mapping \( E_w : \mathcal{L}(M) \to \mathbb{C}^d \). Taking orthogonal complements we see that

\[
\text{ran}(S - \alpha) = \mathcal{L}(M) \cap \ker E_\alpha, \quad \alpha \in \text{hol } (M).
\]

Thus (A) and (B) of Theorem 1.1 hold. Moreover, [16, Theorem 2.1] and its proof imply that there is a constant invertible \( d \times d \) matrix \( T \) such that

\[
TM(z)T^* = M_0 + \begin{bmatrix} \hat{M}(z) & 0 \\ 0 & 0 \end{bmatrix},
\]

where \( \hat{M}(z) \) is the \( d \times d \) extension of \( M(z) \).
where $M_0$ is a constant self-adjoint $d \times d$ matrix and, if the defect numbers of $S$ are denoted by $l$, $\tilde{M}(z)$ is a generalized Nevanlinna $l \times l$ matrix function which is a $Q$-function for $S$. The theorem below concerns a converse implication. But first we recall the notion of a $Q$-function.

Let $S$ be a simple symmetric operator in a Pontryagin space $\mathcal{K}$ with defect index $(l, l)$. Let $A$ be a self-adjoint extension of $S$ in $\mathcal{K}$ with a nonempty resolvent set $\rho(A)$. Let $\mu \in \rho(A) \setminus \mathbb{R}$ and define a function $\Gamma_\mu : \mathbb{C}^l \to \mathcal{K}$ such that it is a linear bijection from $\mathbb{C}^l$ onto $\ker (S^* - \mu)$. Finally, for $z \in \rho(A)$ define the defect mappings $\Gamma_z : \mathbb{C}^l \to \mathcal{K}$ by

$$
\Gamma_z = (I + (z - \mu)(A - z)^{-1}) \Gamma_\mu, \quad z \in \rho(A).
$$

Then $\Gamma_z$ is a bijection from $\mathbb{C}^l$ onto $\ker (S^* - z)$,

$$
\mathcal{K} = \overline{\text{span}} \{ \Gamma_z \mathbf{c} : z \in \rho(A) \cap (\mathbb{C} \setminus \mathbb{R}), \mathbf{c} \in \mathbb{C}^l \}
$$

and, by the resolvent identity, $\Gamma_w^* \Gamma_z = \Gamma_z^* \Gamma_w^*$, $w, z \in \rho(A)$. A $Q$-function for $S$ is by definition an $l \times l$ matrix function that satisfies the equation

$$
\frac{Q(z) - Q(w)^*}{z - w^*} = \Gamma_w^* \Gamma_z, \quad z, w \in \rho(A).
$$

Clearly, $Q(z)$ depends on the choice of the pair $\{A, \Gamma_z\}$ and if this choice has to be mentioned explicitly we shall say that $Q(z)$ is a $Q$-function for $S$ associated with the pair $\{A, \Gamma_z\}$. $Q(z)$ is uniquely determined up to an additive constant self-adjoint $d \times d$ matrix $Q_0$:

$$
Q(z) = Q_0 - i \text{Im } \mu \Gamma_\mu^* \Gamma_\mu + (z - \mu^*) \Gamma_\mu^* \Gamma_z, \quad Q_0 = Q_0^*.
$$

From [5.1] and the defining relation [5.2] it follows that $Q(z)$ is a generalized Nevanlinna $l \times l$ matrix function with $\kappa$ negative squares where $\kappa$ is the negative index of the Pontryagin space $\mathcal{K}$; in particular $Q(z)^* = Q(z^*)$. $Q$-functions in an indefinite setting were introduced and studied by M.G. Krein and H. Langer in [27, 28].

### 5.2. The following theorem shows that the Nevanlinna pair $\{M(z), N(z)\}$ of matrix polynomials $M(z)$ and $N(z)$ in Theorem [1.1] can be chosen such that $\det N(z) \neq 0$ and such that $N(z)^{-1} M(z)$ is essentially the $Q$-function for $S_\mathcal{B}$. As before, by $L(Q)$ we denote the reproducing kernel space with reproducing kernel given by [5.2]

**Theorem 5.1.** Let $\mathcal{B}$ be a finite dimensional Pontryagin subspace of $\mathbb{C}^d[z]$ for which the equivalent conditions of Theorem [1.1] hold. Denote by $l \in \{1, \ldots, d\}$ the equal defect numbers of the symmetric operator $S_\mathcal{B}$. Let $Q(z)$ be an $l \times l$ matrix $Q$-function for $S_\mathcal{B}$. Then there is a $d \times d$ matrix polynomial $N(z)$ with $\det N(z) \neq 0$ such that $M(z) = N(z) \text{diag}(Q(z), 0)$ is a $d \times d$ matrix polynomial and $\mathcal{B} = N(L(Q) \oplus \{0\})$. In particular, $\{M(z), N(z)\}$ is a Nevanlinna pair of matrix polynomials and $K_{M,N}(z, w)$ is the reproducing kernel of $\mathcal{B}$.

**Proof.** Assume [A] and [B] of Theorem [1.1]. By [B] and Theorem [1.2] there is a $d \times d$ matrix function $W(z)$ with $\det W(z) \neq 0$ such that $\mathcal{B} = W \mathcal{C}$, where $\mathcal{C}$ is a canonical subspace of $\mathbb{C}^d[z]$. As to [A]: Clearly, the symmetric operator $S_\mathcal{B}$ has no eigenvalues and for any subset $\mathcal{M}$ of $\mathbb{C}$ containing more than $d \times \max \{\deg f : f \in \mathcal{B}\}$ elements we have $\cap_{w \in \mathcal{M}} \ker (S_\mathcal{B} - w^*) = \{0\}$ or, equivalently,

$$
\mathcal{B} = \overline{\text{span}} \{ \ker (S_\mathcal{B} - w) : w \in \mathcal{M} \}.
$$
It follows that $\mathcal{S}_\mathfrak{B}$ is simple and hence the defect numbers of $\mathcal{S}_\mathfrak{B}$ coincide—we denote them by $l$—and are equal to the codimension of $\text{ran}\, \mathcal{S}_\mathfrak{B}$, see Lemma 2.4.

We consider two cases: $l = d$ and $l < d$.

(i) $l = d$. Let $Q(z)$ be the $Q$-function for $\mathcal{S}_\mathfrak{B}$ associated with the pair $\{A, \Gamma_x\}$, where $A$ is a self-adjoint extension of $\mathcal{S}_\mathfrak{B}$ and the defect mappings $\Gamma_x$ are defined above with $l = d$. Since $\mathcal{S}_\mathfrak{B}$ is simple, the mapping

$$f \mapsto g$$

with $f(z) = \Gamma^*_x, \Gamma_x \cdot x$, $g(z) = (\Gamma_x \cdot x)(z)$, $x \in \mathbb{C}^d$, $w \in \rho(A)$,

can be extended by linearity to a unitary mapping $U$ from $\mathcal{L}(Q)$ onto $\mathfrak{B}$. That $U$ is isometric follows from

$$\left[\Gamma_x \Gamma^*_x \cdot x, \Gamma^*_x \Gamma_x \cdot y\right]_{\mathcal{L}(Q)} = y^* \Gamma^*_x \Gamma_x \cdot x = y^* \Gamma^*_x, \Gamma_x \cdot x = \left[\Gamma_x \cdot x, \Gamma_x \cdot y\right]_{\mathfrak{B}}, \quad x, y \in \mathbb{C}^d.$$

We claim that $U$ is the operator of multiplication by a $d \times d$ matrix function. To prove the claim we use the equality $\mathfrak{B} = W\mathcal{C}$. Since the degrees of $\mathfrak{C}$ are equal to $d$, the degrees of $\mathcal{C}$ are all $\geq 1$ and hence the $d$ columns of $W(z)$ belong to $\mathfrak{B}$ and are linearly independent over $\mathcal{C}$. We denote by $\Gamma^*_x, W$ the $d \times d$ matrix function defined by

$$\left(\Gamma_x^*, W\right) x = \Gamma_x^* (W x), \quad x \in \mathbb{C}^d, \quad z \in \rho(A).$$

We show that its inverse exists for $z \in \mathcal{M} := \rho(A) \cap \{z \in \mathcal{C} : \det W(z) \neq 0\}$. Suppose there is an $x \in \mathbb{C}^d$ such that $(\Gamma_x^*, W) x = 0$. Then for all $y \in \mathbb{C}^d$

$$0 = \left[\Gamma_x^*, W\right] x, y = \left[\Gamma_x^*, (W x), y\right]_{\mathfrak{B}},$$

hence $W x \in \ker (\mathcal{S}_\mathfrak{B}^* - z^*)^{} = \text{ran} (\mathcal{S}_\mathfrak{B}^* - z) = \mathfrak{B} \cap E_z$. That is, $W(z)x = 0$ and it follows that $x = 0$. This proves that $(\Gamma_x^*, W)^{-1}$ is well defined for all $z \in \mathcal{M}$. We set $N(z) = W(z)(\Gamma_x^*, W)^{-1}$. Clearly, $\det N(z) \neq 0$. We have shown that $U$ coincides with multiplication by $N(z)$ if we have proved that

$$N(z) \Gamma^*_x, \Gamma_x \cdot x = (\Gamma_x, x)(z), \quad x \in \mathbb{C}^d, \quad w \in \rho(A), \quad z \in \mathcal{M},$$

or, equivalently, that with $y(z, w, x) := W(z)^{-1}(\Gamma_x \cdot x)(z) \in \mathbb{C}^d$

$$\Gamma_x^* \Gamma_x \cdot x = \Gamma_x^* \left( W y(z, w, x) \right), \quad x \in \mathbb{C}^d, \quad w \in \rho(A), \quad z \in \mathcal{M}.$$ 

But this equality holds, since $\Gamma^*_x \left( \text{ran} (\mathcal{S}_\mathfrak{B}^* - z) \right) = \{0\}$ and

$$\Gamma_x \cdot x - W y(z, w, x) \in \mathfrak{B} \cap E_z = \text{ran} (\mathcal{S}_\mathfrak{B}^* - z).$$

This completes the proof of the claim that $U$ is multiplication by $N(z)$. It follows from [3] Theorem 1.5.7 and its proof that the formula for the kernel $K(z, w)$ of $\mathfrak{B}$ is given by

$$K(z, w) = N(z) \frac{Q(z) - Q(w)^*}{z - w^*} N(w)^*$$

and hence $\mathfrak{B} = N\mathcal{L}(Q)$.

It remains to show that $M(z) = N(z)Q(z)$ and $N(z)$ are matrix polynomials. Since the elements of the space $\mathfrak{B}$ are polynomials the matrix function $z \mapsto K(z, w)$ is a matrix polynomial, hence the matrix function

$$M(z) - N(z)Q(w)^* = N(z)Q(z) - N(z)Q(w)^* = (z - w^*)K(z, w)N(w)^*$$

holds for all $z \neq w$.
is a matrix polynomial in $z$. Thus if $N(z)$ is a matrix polynomial, then so is $M(z)$.
It remains to show that $N(z)$ is a polynomial. For this we note that the above formula implies that for $x \in \mathbb{C}^d$
$$N(z) \frac{Q(\mu^*) - Q(w)^*}{\mu^* - w^*} x = \frac{(z - \mu^*) K(z, \mu) N(\mu)^* - (z - w^*) K(z, w) N(w)^*}{\mu^* - w^*} x.$$ The right hand side is a matrix polynomial in $z$ and hence it follows from the equality that $N(z)$ is a matrix polynomial if we can show that
$$\mathbb{C}^d = \text{span}\{\Gamma_{\mu}^* \Gamma_{w^*} x : w \in \rho(A) \cap (\mathbb{C} \setminus \mathbb{R}), x \in \mathbb{C}^d\}.$$ To prove this equality we argue by contradiction and suppose it is not true. Then there is a nonzero vector $x \in \mathbb{C}^d$ orthogonal to the set on the right hand side, that is,
$$[\Gamma_{w^*} y, \Gamma_{\mu} x]_\mathcal{B} = 0, \quad w \in \rho(A) \cap (\mathbb{C} \setminus \mathbb{R}), \quad y \in \mathbb{C}^d.$$ Since $S_{\mathcal{B}}$ is simple and $\Gamma_{\mu}$ is injective, we find that $\Gamma_{\mu} x = 0$ and that $x = 0$, which contradicts the choice of the nonzero vector $x$.

(ii) $l < d$. Then $\mathcal{C} = \mathcal{C}_1 \oplus \{0\}$, where $\mathcal{C}_1$ is a canonical subspace of $C^d[z]$ of which the degrees are all $\geq 1$. Using the relation $\mathcal{B} = W (\mathcal{C}_1 \oplus \{0\})$ we equip $\mathcal{C}_1$ with an indefinite inner product that makes $W$ an isomorphism. Then $S_{\mathcal{B}}$ and $S_{\mathcal{C}_1}$ are isomorphic: $S_{\mathcal{C}_1} = W S_{\mathcal{B}} W^{-1}$, hence $S_{\mathcal{C}_1}$ is symmetric and has defect index $(l, l)$. Thus (A) holds and it is not difficult to verify that also (B) holds on $\mathcal{C}_1$. Finally, since $Q(z)$ is the $Q$-function for $S_{\mathcal{B}}$ associated with the pair $\{A, \Gamma_z\}$, $Q(z)$ is the $Q$-function for $S_{\mathcal{C}_1}$ associated with the pair $\{W^{-1} AW, W^{-1} \Gamma_z\}$. This all shows that we may apply part (i) of this proof (with $W(z) = I_l$): There exists an $l \times l$ matrix polynomial $N_1(z)$ with $\det N_1(z) \neq 0$ such that $N_1(z) Q(z)$ is an $l \times l$ matrix polynomial and $\mathcal{C}_1 = N_1 \mathcal{L}(Q)$. It follows that if
$$N(z) = W(z) \text{diag} (N_1(z), I_{d-l}),$$ then $\det N(z) \neq 0$, $N(z) \text{diag} (Q(z), 0)$ is a $d \times d$ matrix polynomial and $\mathcal{B} = N (\mathcal{L}(Q) \oplus \{0\})$. \hfill $\square$

6. Corollaries and examples

In the next corollary we extend Theorem 1.1 to finite dimensional Pontryagin spaces of rational vector functions. A rational Nevanlinna kernel is a kernel of the form $K_{M,N}(z,w)$ as in (1.3), in which $N(z)$ and $M(z)$ are rational $d \times d$ matrix functions satisfying (1.4) and (1.5).

**Corollary 6.1.** Let $\mathcal{B}$ be a finite dimensional Pontryagin space of rational $d \times 1$ vector functions. Denote by $S_{\mathcal{B}}$ the operator of multiplication by the independent variable in $\mathcal{B}$ and by $E_\alpha$ the operator of evaluation at a point $\alpha \in \mathbb{C}$. Then the reproducing kernel of $\mathcal{B}$ is a rational Nevanlinna kernel if and only if the following two conditions hold:

(a) The operator $S_{\mathcal{B}}$ is symmetric in $\mathcal{B}$.
(b) For some $\alpha \in \mathbb{C}$ we have $\text{ran}(S_{\mathcal{B}} - \alpha) = \mathcal{B} \cap \ker E_\alpha$.

**Proof.** Let $q(z)$ be the monic scalar polynomial of minimal degree such that $\mathcal{B}' := \{q(z) f(z) : f \in \mathcal{B}\}$ consists of polynomials. Equip $\mathcal{B}'$ with the Pontryagin space inner product that makes the mapping $q : \mathcal{B} \to \mathcal{B}'$ of multiplication by $q(z)$ a unitary mapping. Then items (A) and (B) of Theorem 1.1 hold for $\mathcal{B}'$. Hence $\mathcal{B}'$
has a polynomial reproducing Nevanlinna kernel \( K_{N,M}(z,w) \). It follows that \( \mathcal{B} \) has reproducing kernel \( K_{N/q,M/q}(z,w) \). □

**Corollary 6.2.** Let \( (\mathcal{B},[\cdot,\cdot]_{\mathcal{B}}) \) be a finite dimensional Pontryagin subspace of \( \mathbb{C}^d[z] \) whose reproducing kernel is a Nevanlinna kernel determined by a generalized Nevanlinna pair. Let \( J \) be a fundamental symmetry on \( \mathcal{B} \). Then the Hilbert space \( (\mathcal{B},[J\cdot,\cdot]_{\mathcal{B}}) \) has a Nevanlinna kernel determined by a Nevanlinna pair if and only if \( S_{\mathcal{B}} \) is symmetric in this space.

The corollary follows from Theorem 1.1, because condition (B) is independent of the topology on \( \mathcal{B} \).

**Example 6.3.** Consider the subspace \( \mathcal{B} \) of \( \mathbb{C}^2[z] \) spanned by the columns of the matrix

\[
B(z) = \begin{bmatrix}
1 & z & z^2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

and equipped with the inner product \( [\cdot,\cdot]_{\mathcal{B}} \) so that

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

is the Gram matrix associated with \( B(z) \): \( G = [B,B]_{\mathcal{B}} \). The spectral decomposition of \( G \) is \( G = UJU^* \) with unitary matrix

\[
U = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & 0 & \sqrt{2} \\
-1 & 1 & 0 \\
0 & \sqrt{2} & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and \( J = \text{diag}(-1,1,1,1) \).

It follows that \( (\mathcal{B},[\cdot,\cdot]_{\mathcal{B}}) \) is a Pontryagin space with positive index 3 and negative index 1. The equality \( [BU, BU]_{\mathcal{B}} = J \) defines a fundamental decomposition of \( \mathcal{B} \) with corresponding fundamental symmetry \( J \) determined by \( JBU = BUJ \). In the Hilbert space inner product \( [\cdot,\cdot]_{\mathcal{J}} := [J\cdot,\cdot]_{\mathcal{B}} \) we have \( [BU, BU]_{\mathcal{J}} = J^2 = I_n \) and hence \( [B,B]_{\mathcal{J}} = I_n \). The operator \( S_{\mathcal{B}} \) is symmetric in the Pontryagin space \( (\mathcal{B},[\cdot,\cdot]_{\mathcal{J}}) \), but not in the Hilbert space \( (\mathcal{B},[\cdot,\cdot]_{\mathcal{J}}) \). Since \( \mathcal{B} \) is a canonical subspace of \( \mathbb{C}^2[z] \), Theorem 1.2 implies that Theorem 1.1 (B) holds in \( \mathcal{B} \).

Hence, according to Theorem 1.1, the Pontryagin space \( (\mathcal{B},[\cdot,\cdot]_{\mathcal{J}}) \) has a reproducing Nevanlinna kernel, whereas the reproducing Hilbert space \( (\mathcal{B},[\cdot,\cdot]_{\mathcal{J}}) \) does not have a reproducing Nevanlinna kernel. □

**Corollary 6.4.** Let \( \mathcal{B} \) be a finite dimensional Pontryagin subspace of \( \mathbb{C}^d[z] \) whose reproducing kernel is a Nevanlinna kernel. Let \( \mathcal{B}_0 \) be a Pontryagin subspace of \( \mathcal{B} \). Then the reproducing kernel of \( \mathcal{B}_0 \) is a Nevanlinna kernel if and only if for some \( \alpha \in \mathbb{C} \) we have \( \text{ran}(S_{\mathcal{B}_0} - \alpha) = \mathcal{B} \cap \ker E_\alpha \).

The corollary follows from Theorem 1.1 because the hypothesis implies that \( S_{\mathcal{B}_0} \), being a subset of \( S_{\mathcal{B}} \), is symmetric in \( \mathcal{B}_0 \), that is, that (A) holds for \( S_{\mathcal{B}_0} \).

We end the paper with two examples.
Example 6.5. Consider the space \( \mathcal{B} \) with reproducing kernel

\[
K(z, w) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -w^* \\ -1 & -z & 0 \end{bmatrix}.
\]

We show that, even though \( \det K(z, w) \equiv 0 \), the kernel is a Nevanlinna kernel. We follow the proof of the first part of Theorem 5.1 and construct two Nevanlinna pairs that determine \( K(z, w) \). The space \( \mathcal{B} \) is spanned by the columns of the \( 3 \times 4 \) matrix polynomial

\[
B(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \end{bmatrix}.
\]

It follows that \( \mathcal{B} \) is a canonical subspace of \( \mathbb{C}^3[z] \) with degrees 1, 1 and 2. The Gram matrix associated with \( B(z) \) is given by

\[
G = [B, B]_{\mathcal{B}} = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix},
\]

hence \( \mathcal{B} \) is a Pontryagin space with positive and negative index 2. The operator of multiplication by \( z \) on \( \mathcal{B} \) is given by

\[
S_{\mathcal{B}} = \left\{ \begin{bmatrix} 0 \\ a \\ 0 \\ 0 \\ b \\ 0 \\ 0 \\ c \\ 0 \\ d \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\}.
\]

It is easy to see that (A) and (B) of Theorem 1.1 are satisfied. The defect index of \( S_{\mathcal{B}} \) is \((3, 3)\). The \( Q \)-function of \( S_{\mathcal{B}} \) associated with the self-adjoint extension

\[
A = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ b \\ 0 \\ c \\ 0 \\ d \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\}
\]

of \( S_{\mathcal{B}} \) (which is multi-valued and has a nonempty resolvent set) and the defect mappings \( \Gamma_z : \mathbb{C}^3 \to \ker(S^* - z) \) defined by

\[
\Gamma_z = (I + (z - i)(A - z)^{-1}) \Gamma_1 = B \begin{bmatrix} i/z & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix},
\]

is

\[
Q(z) = Q_0 + \begin{bmatrix} 0 & 1/z & iz \\ 1/z & 0 & 0 \\ -iz & 0 & 0 \end{bmatrix}, \quad Q_0 = Q^*_0.
\]

We find that \( X(z) = Y(z)Q(z) \) and \( Y(z) = (\Gamma_z^* \cdot I)^{-1} \) form a full generalized Nevanlinna pair of matrix polynomials:

\[
X(z) = Y(z)Q_0 + \begin{bmatrix} -i & 0 & 0 \\ iz & 0 & 0 \\ 0 & -iz & 0 \end{bmatrix} \quad \text{and} \quad Y(z) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -iz & 0 \\ -iz & 0 & 0 \end{bmatrix},
\]

such that \( K(z, w) = K_{X,Y}(z, w) \) and \([X(z) \, Y(z)]\) is row reduced with Forney indices 1, 1 and 2.
The $Q$-function associated with the self-adjoint operator extension $A$ of $S_B$ and defect mappings $\Gamma_z$ defined by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_z = \begin{bmatrix} -1/z^2 & 0 & 0 \\ -1/z & 0 & 0 \\ 0 & 1/z & 0 \\ 0 & -(z-i)/z^2 & i/z \end{bmatrix}$$

is given by

$$Q(z) = Q_1 + \begin{bmatrix} 0 & -iz^2 + z - i & -1 \\ iz^2 + z + i & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad Q_1 = Q_1^*.$$  

Again we find that $M(z) = N(z)Q(z)$ and $N(z) = (\Gamma_z^* \cdot I)^{-1}$ are matrix polynomials:

$$M(z) = N(z)Q_1 + \begin{bmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N(z) = \begin{bmatrix} 0 & -iz & z + i \\ 0 & 0 & -iz \\ iz^2 & 0 & 0 \end{bmatrix}$$

which form a full generalized Nevanlinna pair such that $K(z, w) = K_{M,N}(z, w)$ and $[M(z) \quad N(z)]$ is row reduced with Forney indices 1, 1 and 2.

Two generalized Nevanlinna pairs $\{X(z), Y(z)\}$ and $\{M(z), N(z)\}$ of $d \times d$ matrix polynomials define the same Nevanlinna kernel if one is a $J$-unitary transformation of the other, that is, if they are connected via the formulas

$$M(z) = X(z)A + Y(z)C, \quad N(z) = X(z)B + Y(z)D,$$

where $A$, $B$, $C$ and $D$ are constant $d \times d$ matrices such that if we set

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & iI_d \\ -iI_d & 0 \end{bmatrix},$$

then $U$ is $J$-unitary: $UJU^* = J$. The pairs $\{X(z), Y(z)\}$ and $\{M(z), N(z)\}$ in Example 6.5 (with arbitrary constant self-adjoint matrices $Q_0$ and $Q_1$) are connected via a $J$-unitary transformation. The following example shows that the converse of the foregoing statement does not hold.

**Example 6.6.** Consider the Nevanlinna pair $\{X(z), Y(z)\}$ given by

$$X(z) = \text{diag}(z, 0, z^2), \quad Y(z) = \text{diag}(0, 0, z).$$

Then the space $\mathcal{B}$ with reproducing kernel $K_{X,Y}(z, w) = \text{diag}(0, 0, zw^*)$ is a 1-dimensional Hilbert space: it is spanned by $B(z) = [0 \quad 0 \quad z]^T$ and the corresponding Gram matrix is $G = [B, B]_\mathcal{B} = 1$. Note that $\text{det} X(z) \equiv 0$, $\text{det} Y(z) \equiv 0$ and $\mathbf{P}(z) := [X(z) \quad Y(z)]$ does not have full rank at $z = 0$: $\mathbf{P}(0) = 0$. We show that the pair $\{X(z), Y(z)\}$ can be replaced by a Nevanlinna pair $\{M(z), N(z)\}$ such that $\text{det} N(z) \neq 0$.

Since $\mathbf{P}(z)$ does not have full rank for all $z \in \mathbb{C}$, to calculate the Forney indices we must first apply Lemma 2.4. We write $\mathbf{P}(z)$ as $\mathbf{P}(z) = G(z)\mathbf{T}(z)$ with

$$G(z) = \begin{bmatrix} 0 & 0 & z \\ 0 & z & 0 \\ z & 0 & 0 \end{bmatrix}, \quad \mathbf{T}(z) = \begin{bmatrix} 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. $$
Since $\mathbf{T}(z)$ has full rank for all $z \in \mathbb{C}$ and is row reduced, the Forney indices of $\mathbf{P}(z)$ are those of $\mathbf{T}(z)$ and they are $\mu_1 = 1$, $\mu_2 = \mu_3 = 0$. This fits in well with the observations after Theorem 5.1 and write $\mathcal{B} = W(\mathcal{C}_1 \oplus \{0\})$ with

$$W(z) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ z & 0 & 0 \end{bmatrix}$$

and $\mathcal{C}_1 = \mathbb{C}$. We make the multiplication operator $W$ an isometry when $\mathbb{C}$ is equipped with the Euclidean inner product. Then $\mathcal{S}_{\mathcal{C}_1} = \{\{0,0\}\}$ is symmetric. The defect subspaces $\ker(S_{\mathcal{C}_1}^* - z)$ all coincide with $\mathbb{C}$ and $A$ is a self-adjoint extension of $\mathcal{S}_{\mathcal{C}_1}$ if and only if $A = A_m$, the operator of multiplication by $m$, $m \in \mathbb{R}$, or $A = A_{\text{rel}} = \{\{0, c\} : c \in \mathbb{C}\}$. Since $\mathcal{C}_1$ is a Hilbert space, all self-adjoint operators and relations have a non-empty resolvent set. Choose $\mu \in \mathbb{C} \setminus \mathbb{R}$, $\gamma \in \mathbb{C} \setminus \{0\}$ and define $\Gamma_\mu : \mathbb{C} \to \ker(S_{\mathcal{C}_1}^* - \mu) = \mathbb{C}$ by $\Gamma_\mu(x) = \gamma x$, $x \in \mathbb{C}$. Then the $Q$-function $q(z)$ of $\mathcal{S}_{\mathcal{C}_1}$ associated with $\{A, \Gamma_\mu\}$ is given by

$$q(z) = q_0 + \begin{cases} \frac{|m - \mu|^2 |\gamma|^2}{m - z} & \text{if } A = A_m, \\ |\gamma|^2 z & \text{if } A = A_{\text{rel}}, \end{cases}$$

where $q_0$ is an arbitrary real number, and

$$(\Gamma_\mu^*, 1)^{-1} = \begin{cases} \frac{m - z}{\gamma^*(m - \mu^*)} & \text{if } A = A_m, \\ \frac{1}{\gamma^*} & \text{if } A = A_{\text{rel}}. \end{cases}$$

We find that $K_{X,Y}(z, w) = K_{M,N}(z, w)$ with matrix polynomials

$$M(z) = N(z) \begin{bmatrix} q(z) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N(z) = W(z) \begin{bmatrix} (\Gamma_\mu^*, 1)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$


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