The medians are special

A median of a triangle is a line segment that connects a vertex of the triangle to the midpoint of the opposite side. The three medians of a triangle interact nicely with each other to yield the following properties:

(a) The medians intersect in a point interior to the triangle, called the centroid, which divides each of the medians in the ratio 2 : 1.
(b) The medians form a new triangle, called the median triangle.
(c) The area of the median triangle is $\frac{3}{4}$ of the area of the given triangle in which the medians were constructed.
(d) The median triangle of the median triangle is similar to the given triangle with the ratio of similarity $\frac{3}{4}$.

When we say, as in (b), that “three line segments form a triangle” we mean that there exists a triangle whose sides have the same lengths as the line segments.

![Figure 1](image)

**Figure 1** A “proof” of Properties (b) and (c)

Proving Property (a) is a common exercise. We provide “proofs without words” of Properties (b), (c), and (d) in Figures 1 and 2. Different proofs can be found in [9] and at [17]. Note that Property (b) fails for other equally important triples of cevians of a triangle; for example, as shown in [2], we cannot always speak about a triangle formed by bisectors or altitudes.

History  Before introducing outer analogues of the medians and the median triangle, we reflect on the history of the above properties. Property (a) was proved by Archimedes of Syracuse, as Proposition 14 in Equilibrium of Planes, Book I; see [8], [1, Subsection 10.7.2], or [16, p. 86]. By the 19th century, Property (a) was a common proposition accompanying Euclid’s Elements [4, 14]. Interestingly, in [14] and in several other books of this period, the term “median triangle” of $ABC$ means, in our notation, the triangle $A_{1/2}B_{1/2}C_{1/2}$. Nowadays, this triangle is called the medial triangle of $ABC$.

The first usage of the current meaning of the median triangle that we found is in [11, Ch. XVI, §473]. However, in his 1887 paper [13], Mackay proves Property (b) in his §6 without explicitly stating it. He attributes his §6 to [15], but we could not find it there.

Furthermore, Property (c) appears as [13, §8(c)] and Property (d) as [13, §8(a)]. Mackay points out that [13, §8(a)] is proved in [10] as a solution to a problem proposed in [12]. Finally, Mackay believed that his [13, §8(c)] was new.

The medians are not alone

A median of a triangle is just a special cevian; a cevian is a line segment joining a vertex of a triangle to a point on the opposite side. Are there other triples of cevians from distinct vertices of a triangle that share the essential features of Properties (a), (b), (c), and (d)?

Some natural candidates for such cevians are suggested by the “median grid” already encountered in Figures 1 and 2. In Figures 3 and 4, we show more of this grid with the cevians that in some sense most resemble the medians. The labeling of the points on the line $BC$ in Figure 4 originates from $BC$ being considered as a number line with 0 at $B$ and 1 at $C$. More precisely, for $\rho \in \mathbb{R}$, the point $A_\rho$ is the point on the line $BC$ that satisfies $\overrightarrow{BA_\rho} = \rho \overrightarrow{BC}$. The points on the lines $AB$ and $BC$ are labeled similarly.

As indicated in Figure 3, the cevians in the triple

\[
(BB_{-1/2}, AA_{1/2}, CC_{3/2})
\]

are concurrent, with the understanding that three line segments are concurrent if the lines determined by them are concurrent. In addition to the triple of cevians in (1)
we will consider two more triples of concurrent cevians that are symmetrically placed with respect to the other sides:

\[
(\text{CC}_{-1/2}, \text{BB}_{1/2}, \text{AA}_{3/2}), \quad (\text{AA}_{-1/2}, \text{CC}_{1/2}, \text{BB}_{3/2}).
\] (2)

All three triples are shown in Figure 4.

That the triples in (1) and (2) are really concurrent follows from Ceva’s theorem, which in our notation reads as:

**Ceva’s Theorem** [5, p. 220]. With \( \rho, \sigma, \tau \in \mathbb{R} \), the cevians \( \text{AA}_\rho, \text{CC}_\sigma, \text{BB}_\tau \) are concurrent if and only if

\[
\rho \sigma \tau - (1 - \rho)(1 - \sigma)(1 - \tau) = 0.
\] (3)

Equation (3) defines a surface in \( \rho \sigma \tau \)-space; see Figure 10, below. We call it the *Ceva surface*. It will appear prominently in what follows.
Since the cevians $AA_{-1/2}$, $AA_{3/2}$, $BB_{-1/2}$, $BB_{3/2}$, $CC_{-1/2}$, $CC_{3/2}$ play the leading roles in this note and because of their proximity to the medians on the “median grid,” we call them outer medians. Thus, for example, associated to vertex $A$ we have one median, $AA_{1/2}$, and two outer medians, $AA_{-1/2}$ and $AA_{3/2}$. See Figure 4.

We find it quite remarkable that all four properties of the medians listed in the opening of this note hold for the three triples displayed in (1) and (2), each of which consists of a median and two outer medians originating from distinct vertices.

(A) The median and two outer medians in each of the triples in (1) and (2) are concurrent.

(B) The median and two outer medians in the triples in (1) and (2) form three triangles. We refer to these three triangles as outer median triangles of $ABC$; see Figure 5.

(C) The area of each outer median triangle of $ABC$ is $5/4$ of the area of $ABC$.

(D) For each outer median triangle, one of its outer median triangles is similar to the original triangle $ABC$ with the ratio of similarity $5/4$.

![Figure 5](image1.png) Three outer median triangles of $ABC$

As we have already mentioned, Property (A) follows from Ceva’s theorem. Figures 6 and 7 offer “proofs without words” of Properties (B), (C), and (D).

We point out that the concurrency points $G_a$, $G_b$, $G_c$ (see Figure 4) divide the corresponding outer medians in the ratio $2 : 3$, that is, for example,

$$BG_a : G_aB_{-1/2} = CG_a : G_aC_{3/2} = 2 : 3.$$  

![Figure 6](image2.png) A “proof” of Properties (B) and (C)
Similarly, it can be shown that the concurrency points divide the corresponding medians in the ratio 6:1, for example, $AG_a : A_{1/2}G_a = 6:1$. Computing these ratios is an exercise in vector algebra.

Are the medians and the outer medians alone?

We motivated our study of outer medians by their special position on the “median grid.” However, the above four properties could very well hold for other triples of cevians. Is it then the case that the median triangle and the three outer median triangles are truly special?

The concurrency of three cevians is characterized by equation (3), which was used to justify the claim in Property (A). Next, akin to Property (B) and with no requirement, for the moment, that the cevians be concurrent, we look for a sufficient condition under which three cevians form a triangle.

**Property (B)** We first answer the following question: For which $\rho, \sigma, \tau \in \mathbb{R}$ does there exist a triangle with sides that are congruent and parallel to the cevians $AA_\rho$, $BB_\sigma$ and $CC_\tau$, independent of the triangle $ABC$ in which they are constructed?

With $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$ and $\mathbf{c} = \overrightarrow{AB}$, we have

$$\overrightarrow{AA_\rho} = \mathbf{c} + \rho \mathbf{a}, \quad \overrightarrow{BB_\sigma} = \mathbf{a} + \sigma \mathbf{b} \quad \text{and} \quad \overrightarrow{CC_\tau} = \mathbf{b} + \tau \mathbf{c}.$$  

Then, a necessary and sufficient condition for the existence of a triangle with sides that are congruent and parallel to the line segments $AA_\rho$, $BB_\sigma$, and $CC_\tau$ is that one of the following four vector equations is satisfied:

$$\overrightarrow{AA_\rho} \pm \overrightarrow{BB_\sigma} \pm \overrightarrow{CC_\tau} = \mathbf{0}. \quad (4)$$

We put a special sign $\hat{\pm}$ above the first $\pm$ to be able to trace this sign in the calculations that follow. Substituting $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ in (4), we get

$$(-1 \hat{\pm} 1 + \rho \mp \tau) \mathbf{a} + (-1 \hat{\pm} \sigma \pm 1 \mp \tau) \mathbf{b} = \mathbf{0}. \quad (5)$$

Using the linear independence of $\mathbf{a}$ and $\mathbf{b}$ and choosing both $+$ signs in (4), it follows from (5) that $\rho = \sigma = \tau$. Choosing the first sign in (4) to be $+$ and the second to be $-$, we get that $\rho = -\tau$, $\sigma = 2 - \tau$. Choosing the first sign in (4) to be $-$ and the
second to be +, we get \( \rho = 2 - \sigma, \tau = -\sigma \); and choosing both \(-\) signs in (4), we get \( \sigma = -\rho, \tau = 2 - \rho \). Thus, we have identified four sets of parameters \((\rho, \sigma, \tau)\) for which, independent of \(ABC\), there exists a triangle, possibly degenerate, with sides that are congruent and parallel to the cevians \(AA_\rho, BB_\sigma, \) and \(CC_\tau\):

\[
(\xi, \xi, \xi), \quad (2 - \xi, \xi, -\xi), \quad (-\xi, 2 - \xi, \xi), \quad (\xi, -\xi, 2 - \xi), \quad \xi \in \mathbb{R}.
\] (6)

The only concern here is that the cevians \(AA_\rho, BB_\sigma, \) and \(CC_\tau\) might be parallel. However, the condition for the cevians to be parallel is easily established as follows. Since the vector \(\overrightarrow{CC_\tau}\) is nonzero, we look for \(\lambda, \mu \in \mathbb{R}\) such that

\[
c + \rho \, a = \lambda (b + \tau \, c) \quad \text{and} \quad a + \sigma \, b = \mu (b + \tau \, c).
\] (7)

Substituting \(c = -a - b\) in (7) and using the linear independence of \(a\) and \(b\), we get from the first equation \(\lambda = 1/(\tau - 1)\), \(\rho = 1/(1 - \tau)\) and from the second equation \(\mu = -1/\tau, \sigma = 1 - 1/\tau\). Hence, the line segments \(AA_\rho, BB_\sigma, \) and \(CC_\tau\) are parallel if and only if

\[
\rho = \frac{1}{1 - \xi}, \quad \sigma = 1 - \frac{1}{\xi}, \quad \tau = \xi, \quad \xi \in \mathbb{R} \setminus \{0, 1\}.
\] (8)

Thus, to avoid degeneracy of triangles with cevian sides corresponding to triples in (6) such as, for example, \((-\xi, 2 - \xi, \xi)\), we must exclude the values of the parameter \(\xi\) that solve \(-\xi = 1/(1 - \xi)\). This, in turn, shows that the triples \((\rho, \sigma, \tau)\) for which there exists a non-degenerate triangle with sides that are congruent and parallel to the cevians \(AA_\rho, BB_\sigma, \) and \(CC_\tau\) must belong to one of the following four sets:

\[
\begin{align*}
\mathbb{D} &= \{(\xi, \xi, \xi) : \xi \in \mathbb{R}\}, \\
\mathbb{E} &= \{(2 - \xi, \xi, -\xi) : \xi \in \mathbb{R} \setminus \{-\phi^{-1}, \phi\}\}, \\
\mathbb{F} &= \{(-\xi, 2 - \xi, \xi) : \xi \in \mathbb{R} \setminus \{-\phi^{-1}, \phi\}\}, \\
\mathbb{G} &= \{\xi, -\xi, 2 - \xi : \xi \in \mathbb{R} \setminus \{-\phi^{-1}, \phi\}\},
\end{align*}
\]

where \(\phi = (1 + \sqrt{5})/2\) denotes the golden ratio.

The diagonal of the \(\rho\sigma\tau\)-space provides a geometric representation of the set \(\mathbb{D}\). The other three sets are represented by straight lines with two points removed. All four lines are shown in Figure 10, together with the Ceva surface.

**Generalized median and outer median triangles** As we have just seen, the cevians associated with the triples in the sets \(\mathbb{D}, \mathbb{E}, \mathbb{F}, \) and \(\mathbb{G}\) are guaranteed to form triangles; that is, they satisfy a property analogous to Property (B). The most prominent representatives of triangles originating from the sets \(\mathbb{D}, \mathbb{E}, \mathbb{F}, \) and \(\mathbb{G}\) are the median and outer median triangles, which all correspond to the value \(\xi = 1/2\). Therefore, for a fixed \(\xi\), the triangle associated with the triple \((\xi, \xi, \xi)\) in \(\mathbb{D}\) we call \(\xi\)-median triangle, and the triangles associated with the corresponding triples in \(\mathbb{E}, \mathbb{F}, \) and \(\mathbb{G}\) we call \(\xi\)-outer median triangles. In Figures 8 and 9, we illustrate these triangles with \(\xi = 1/\phi\), the reciprocal of the golden ratio.

Next, we explore whether the \(\xi\)-median and \(\xi\)-outer median triangles have properties analogous to (C) and (D).
Property (C) First, we recall two classical formulas, which seem to be custom made for our task. 

*Heron’s formula* [5, 1.53], gives the square of the area of a triangle, $\Delta^2$, in terms of its sides $a, b, c$:

$$\Delta^2 = s(s-a)(s-b)(s-c), \quad \text{where } s = \frac{1}{2}(a+b+c).$$

Substituting $s$ and simplifying yields

$$\Delta^2 = \frac{1}{16}\left(2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)\right).$$

*Stewart’s theorem* [6, Section 1.2, Exercise 4], gives the square of the length of a cevian in terms of the squares of the sides of $ABC$:

$$(AA_\rho)^2 = \rho(\rho - 1)a^2 + \rho b^2 + (1 - \rho)c^2.$$  

Similar formulas hold for $(BB_\rho)^2$ and $(CC_\rho)^2$. In matrix form, these three equations
are:
\[
\begin{pmatrix}
(CC_\tau)^2 \\
(BB_\sigma)^2 \\
(AA_\rho)^2
\end{pmatrix} =
\begin{bmatrix}
\tau & 1 - \tau & \tau(\tau - 1) \\
1 - \sigma & \sigma(\sigma - 1) & \sigma \\
\rho(\rho - 1) & \rho & 1 - \rho
\end{bmatrix}
\begin{bmatrix}
a^2 \\
b^2 \\
c^2
\end{bmatrix}.
\] (9)

We denote the $3 \times 3$ matrix in (9) by $M(\rho, \sigma, \tau)$. The idea of using this matrix is due to Griffiths [7]. It was further explored in [3, Section 3].

Now it is clear how to proceed to verify the property analogous to (C): use triples from the sets $D$, $E$, $F$, and $G$ to get expressions for the squares of the corresponding cevians, substitute these expressions in Heron’s formula, and simplify. However, this involves simplifying an expression with 36 additive terms, quite a laborious task for a human but a perfect challenge for a computer algebra system like Mathematica. We first define Heron’s formula as a Mathematica function (we call it HeronS) operating on the triples of squares of the sides of a triangle and producing the square of the area:

\[
\text{In}[1]:= \text{HeronS}[[x_-, y_-, z_-]] := \frac{1}{16} (2 (x*y + y*z + z*x) - (x^2+y^2+z^2))
\]

Next, we define in Mathematica the matrix function $M$ as in (9):

\[
\text{In}[2]:= M[[\rho_-, \sigma_-, \tau_-]] := \{ \{ \tau, 1 - \tau, -(1 - \tau) * \tau \}, \\
\{ 1 - \sigma, -(1 - \sigma) * \sigma, \sigma \}, \\
\{ -(1 - \rho) * \rho, \rho, 1 - \rho \} \}
\]

To verify the property analogous to (C) for $\xi$-median triangles, we put the newly defined functions in action by calculating the ratio between the squares of the area of the $\xi$-median triangle and the original triangle. Mathematica’s answer is instantaneous:

\[
\text{In}[3]:= \text{Simplify}[	ext{HeronS}[M[[\xi, \xi, \xi]].\{x,y,z\}]/\text{HeronS}[[x,y,z]]]
\]

\[
\text{Out}[3] = (1 - \xi + \xi^2)^2
\]

This “proves” that the ratio of the areas depends only on $\xi$, and that the ratio is exactly $1 - \xi + \xi^2$. Further, for one of the $\xi$-outer median triangles, we have

\[
\text{In}[4]:= \text{Simplify}[	ext{HeronS}[M[[2-\xi, \xi, -\xi]].\{x,y,z\}]/\text{HeronS}[[x,y,z]]]
\]

\[
\text{Out}[4] = (1+\xi-\xi^2)^2
\]

“proving” that the area of the triangle formed by the cevians $AA_{2-\xi}$, $BB_{\xi}$, $CC_{-\xi}$ is $|1 + \xi - \xi^2|$ of the area of the original triangle $ABC$. The other two $\xi$-outer median triangles yield the same ratio. In summary, Mathematica has confirmed that the $\xi$-median and the three $\xi$-outer median triangles all have the property analogous to (C).

**Property (D)** The verification of the property analogous to (D) is simpler. For a $\xi$-median triangle, following [7], we just need to calculate the square of the matrix $M(\xi, \xi, \xi)$, which turns out to be $(1 - \xi + \xi^2)^2 I$. This confirms that the $\xi$-median triangle of the $\xi$-median triangle is similar to the original triangle with the ratio of similarity $1 - \xi + \xi^2$.

Similarly, for a $\xi$-outer median triangle corresponding to a triple in $E$, we calculate the square of the matrix $M(2 - \xi, \xi, -\xi)$, which turns out to be $(1 + \xi - \xi^2)^2 I$; this confirms that one of the $\xi$-outer median triangles of this $\xi$-outer median triangle is
similar to the original triangle with the ratio of similarity $|1 + \xi - \xi^2|$. In contrast, for a $\xi$-outer median triangle corresponding to a triple in $F$, to get a triangle similar to the original triangle we need to calculate its $\xi$-outer median triangle corresponding to a triple in $G$. This amounts to multiplying the matrices

$$M(\xi, -\xi, 2 - \xi)M(-\xi, 2 - \xi, \xi) = (1 + \xi - \xi^2)^2 I.$$  

Likewise, for a $\xi$-outer median triangle corresponding to a triple in $G$, we calculate its $\xi$-outer median triangle corresponding to a triple in $F$ and obtain the same result.

**Concurrency comes to the rescue**  All these calculations indicate that, after all, the median and outer median triangles are facing stiff competition from their $\xi$-triangles generalizations. However, property (A) comes to the rescue of the median and outer median triangles at this point. We want the triples of cevians corresponding to the triples in $D$, $E$, $F$, and $G$ to be concurrent as well. So which of these triples satisfy Ceva’s condition (3)? Or, geometrically, what is the intersection of the lines and the Ceva surface in Figure 10? First, we substitute $\rho = \sigma = \tau = \xi$ in (3), which yields $\xi^3 - (1 - \xi)^3 = 0$, whose only real solution is $\xi = 1/2$. The corresponding cevians are the medians. To intersect $E$ with the Ceva surface, we substitute $(2 - \xi, \xi, -\xi)$ in (3), obtaining $-\xi^2(2 - \xi) - (1 + \xi)(\xi - 1)(1 - \xi) = 0$, which is equivalent to $(\xi - \phi)(\xi + \phi^{-1})(2\xi - 1) = 0$. Since $\xi \not\in \{\phi, -\phi^{-1}\}$, the only solution is $\xi = 1/2$.

**Figure 10**  The sets $D$, $E$, $F$ and $G$ and the Ceva surface
yielding the “outer median triple” \((3/2, 1/2, -1/2)\). Intersecting \(F\) with the Ceva surface gives \((-1/2, 3/2, 1/2)\) and intersecting \(G\) with the Ceva surface results in \((1/2, -1/2, 3/2)\). Consequently, the only triples in \(D, E, F,\) and \(G\) which correspond to concurrent cevians are the “median triple” and the three “outer median triples.”

There is only a slight weakness in our argument above. In identifying the sets \(D, E, F,\) and \(G\), we assumed that the triangles formed by the corresponding cevians have sides that are parallel to the cevians themselves. In [3], we proved that the only cevians \(AA_\rho, BB_\sigma, CC_\tau\) that form triangles and with \((\rho, \sigma, \tau)\) not included in the sets \(D, E, F,\) and \(G\) are parallel cevians, that is the cevians \(AA_\rho, BB_\sigma,\) and \(CC_\tau\), where \(\rho, \sigma, \tau\) satisfy (8) with the additional restriction

\[ \xi \in (-\phi, -\phi^{-1}) \cup (\phi^{-2}, \phi^{-1}) \cup (\phi, \phi^2). \]

As it turns out, the properties analogous to (C) and (D) do not hold for triangles formed by such cevians. In conclusion, indeed, along with the medians and the median triangle, the outer medians and their outer median triangles are unique in satisfying all four properties analogous to those from the beginning of our note.

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**Summary** We define the notions of outer medians and outer median triangles. We show that outer median triangles enjoy similar properties to that of the median triangle.
ÁRPÁD BÉNYI is a Professor in the Department of Mathematics at Western Washington University, located in beautiful Bellingham, WA. Previously he held a Visiting Assistant Professorship at University of Massachusetts, Amherst. His main research interests are in harmonic analysis and its connections to partial differential equations and probability theory. However, his favorite activity is playing with his two young boys, Alexander and Sebastian.

BRANKO ČURGUS received his Ph.D. in mathematics in 1985 from the University of Sarajevo, in the former Yugoslavia. His Ph.D. research was done under the advisement of Prof. Heinz Langer at the Technical University Dresden, in the former German Democratic Republic. Since 1987 he has been enjoying life, teaching and researching mathematics at Western Washington University in Bellingham.

—Lee Sallows