AN ALEKSANDROV-TYPE ESTIMATE FOR A PARABOLIC MONGE-AMPÈRE EQUATION

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ABSTRACT. A classical result of Aleksandrov allows one to estimate the size of a convex function u at a point x in a bounded domain Ω in terms of the distance from x to the boundary of Ω if $\int_\Omega \det D^2u \ dx < \infty$. This estimate plays a prominent role in the existence and regularity theory of the Monge-Ampère equation. Jerison proved an extension of Aleksandrov's result that provides a similar estimate, in some cases for which this integral is infinite. Gutiérrez and Huang proved a variant of the Aleksandrov estimate, relevant to solutions of a parabolic Monge-Ampère equation. In this paper, we prove Jerison-like extensions to this parabolic estimate.

1. INTRODUCTION

In studying the regularity and existence of weak solutions (in the sense of Aleksandrov) to the Dirichlet problem for the Monge-Ampère equation:

$$\begin{aligned} \det D^2 u &= \mu \text{ in } \Omega, \\ u|_{\partial\Omega} &= g, \end{aligned} \tag{1}$$

where μ is a Borel measure on the convex domain Ω and $g \in C(\partial\Omega)$, the following estimate of Aleksandrov plays a critical role. For its applications to this problem, see, for example, [12], [3], and [6]. A variant of this estimate appears in [2].

Theorem 1. *[Aleksandrov's estimate] Let Ω be a bounded convex domain in $\mathbb{R}^n$, and let $u \in C(\bar{\Omega})$ be convex, with $u = 0$ on $\partial\Omega$. Then for all $x \in \Omega$,*

$$|u(x)|^n \leq C_n (diam\, \Omega)^{n-1}\, dist(x, \partial\Omega)\, Mu(\Omega), \tag{2}$$

where C_n is a dimensional constant and Mu is the Monge-Ampère measure associated to u.

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This estimate allows one to estimate the size of u at a point x in terms of the distance from x to the boundary of the domain. However, if u is such that $Mu(\Omega) = \infty$ (which can occur if $|Du| \to \infty$ at $\partial\Omega$), (2) does not give any information about the size of $u(x)$. Jerison, in [10], extended this inequality, using an affine-invariant normalized distance to the boundary, to an estimate (Theorem 2) that is useful even if $Mu(\Omega) = \infty$, provided Mu does not blow up too quickly at the boundary. This result allows for a Caffarelli-style regularity theory for such problems, provided Mu satisfies a technical requirement, weaker than the doubling condition, on the cross-sections of u; see [10] and [7].

The parabolic Monge-Ampère operator $u_t \det D_x^2 u$ was introduced in [11]. It is related to the problem of deformation of surfaces by Gauss curvature (see [13]). This operator is also considered in the following works: [16], [8], [5], [4], [14], [9], and [15].

In studying entire solutions of the parabolic Monge-Ampère equation $-u_t \det D_x^2 u = 1$, Gutiérrez and Huang ([8]) extended the Aleksandrov estimate (Theorem 1) to parabolically convex functions on bounded bowl-shaped domains. This estimate again degenerates when the parabolic Monge-Ampère measure associated to u of the entire domain is infinite. The purpose of this note is to extend the estimates of Jerison to the parabolic setting. These estimates are given below in Lemma 4 and Theorem 4. Because Jerison's estimates allow for a regularity theory for problem (1) when $\mu(\Omega) = \infty$, it is our hope that the estimates presented here will allow one to deduce regularity properties of parabolically convex solutions of the Dirichlet problem:

$$\begin{aligned} -u_t \det D^2 u &= f \text{ in } E \\ u|_{\partial_p E} &= g, \end{aligned}$$

where $f \geq 0$ may fail to be in $L^1(E)$, $E \subset \mathbb{R}^{n+1}$ is bowl-shaped, and $\partial_p E$ is the parabolic boundary of E. This would extend the regularity theory found in [5], [4], [14] and [15], all of which assume that f is bounded.

2. Preliminaries

We begin this section by reviewing the basic theory of weak or generalized solutions, in the Aleksandrov sense, to the (elliptic) Monge-Ampère equation. Proofs of these results and historical notes indicating their original sources can be found in the books [1] and [6].

Given $u : \Omega \to \mathbb{R}$ we recall that the normal mapping (or subgradient) of u is defined by

$$\partial u(x_0) = \{p \in \mathbb{R}^n : u(x) \geq u(x_0) + p \cdot (x - x_0), \ \forall\, x \in \Omega\};$$

and if $E \subset \Omega$, then we set $\partial u(E) = \bigcup_{x \in E} \partial u(x)$. Note that the normal map of u at a point x_0 is the set of points p which are normal vectors for supporting hyperplanes to the graph of u at x_0.

If Ω is open and $u \in C(\Omega)$ then the family of sets

$$S = \{E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable}\}$$

is a Borel σ–algebra. The map $Mu : S \to \bar{\mathbb{R}}$ defined by $Mu(E) = |\partial u(E)|$ (where $|S|$ indicates the Lebesgue measure of the set S) is a measure, finite on compact subsets, called the Monge–Ampère measure associated with the function u. The convex function u is a weak (Aleksandrov) solution of $\det D^2 u = \nu$ if the Monge–Ampère measure Mu associated with u equals the Borel measure ν.

We use the notation $B_r(y)$ for the open Euclidean ball of radius r with center y. The dimension of $B_r(y)$ should be clear from context.

Definition 1. *A convex domain $\Omega \subset \mathbb{R}^n$ with center of mass at the origin is said to be normalized if $B_{\alpha_n}(0) \subset \Omega \subset B_1(0)$, where $\alpha_n = n^{-3/2}$.*

The following lemma allows us to carry out our analysis in a normalized setting. It is a consequence of a result of John on ellipsoids of minimum volume. See Section 1.8 of [6] and its references for more detail.

Lemma 1. *If Ω is a bounded convex domain, there exists an affine transformation T such that $T(\Omega)$ is normalized.*

We now introduce the normalized distance to the boundary used by Jerison in [10].

Definition 2. *Let $\Omega \subset \mathbb{R}^n$ be bounded, open and convex. The normalized distance from $x \in \Omega$ to the boundary of Ω is*

$$\delta(x, \Omega) = \min \left\{ \frac{|x - x_1|}{|x - x_2|} : x_1,\, x_2 \in \partial\Omega \text{ and } x,\, x_1,\, x_2 \text{ are collinear} \right\}.$$

The most important properties of this distance for our purposes are summarized in the following lemma.

Lemma 2. *Let Ω be a bounded convex domain.*

(a) If T is an invertible affine transformation on $\mathbb{R}^n$, then

$$\delta(x, \Omega) = \delta(Tx, T(\Omega)).$$

(b) If Ω is normalized, $\delta(x, \Omega)$ is equivalent to $\operatorname{dist}(x, \partial\Omega)$, i.e. there exist constants C_1 and C_2 (depending only on the dimension) such that

$$C_1 \delta(x, \Omega) \le \operatorname{dist}(x, \partial\Omega) \le C_2 \delta(x, \Omega)$$

for all $x \in \Omega$, where dist *is the Euclidean distance.*

(c) For all $x \in \Omega$, $\operatorname{dist}(x, \partial\Omega) \le \operatorname{diam}(\Omega)\delta(x, \Omega)$.

We now state Jerison's estimates. The first (Lemma 3) is Lemma 7.2 in [10]. Estimate (3) is similar to Aleksandrov's estimate (2), with the normalized notion of distance replacing the standard one, and the Lebesgue measure of Ω replacing the diameter term.

Lemma 3. *Let Ω be an open convex set and suppose $u \in C(\overline{\Omega})$ is convex and zero on $\partial\Omega$. Then, for all $x \in \Omega$,*

$$|u(x)|^n \leq C\delta(x,\Omega)|\Omega|Mu(\Omega) \tag{3}$$

where C is a constant depending only on the dimension.

Notice that the estimate (3) gives no information when $Mu(\Omega) = \infty$. If this is the case, Mu must blow up near $\partial\Omega$, but this is precisely where $\delta(\cdot,\Omega)$ is small. As a consequence, the estimate in the next result (Lemma 7.3 in [10]) may be meaningful.

Theorem 2. *Let Ω be bounded, open, convex and normalized, and suppose $u \in C(\overline{\Omega})$ is convex and zero on $\partial\Omega$. For each $\epsilon \in (0,1]$, there exists a constant $C(n,\epsilon)$ such that*

$$|u(x_0)|^n \leq C(n,\epsilon)\delta(x_0,\Omega)^\epsilon \int_\Omega \delta(x,\Omega)^{1-\epsilon}\, dMu(x) \tag{4}$$

for all $x_0 \in \Omega$.

We now introduce some terminology and notation for the parabolic problem. Let $D \subset \mathbb{R}^{n+1}$ and let $t \in \mathbb{R}$. Then define

$$D(t) = \{x \in \mathbb{R}^n : (x,t) \in D\}.$$

Definition 3. *The domain D is said to be bowl-shaped if $D(t)$ is convex for every t and $D(t_1) \subset D(t_2)$ whenever $t_1 \leq t_2$. If D is bounded, let $t_0 = \inf\{t : D(t) \neq \emptyset\}$. Then the parabolic boundary of D is defined to be*

$$\partial_p D = (\bar{D}(t_0) \times \{t_0\}) \cup \left(\bigcup_{t\in\mathbb{R}} (\partial D(t) \times \{t\}) \right).$$

For a bowl-shaped domain D we define the set D_{t_0} to be $D_{t_0} = D \cap \{(x,t) : t \leq t_0\}$.

Definition 4. *A function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $u = u(x,t)$, is called parabolically convex (or convex-monotone) if it is continuous, convex in x and non-increasing in t.*

We now define the parabolic normal map and parabolic Monge-Ampère measure. As in the elliptic case, this will lead to the notion of weak solution for this operator. Let $D \subset \mathbb{R}^{n+1}$ be an open, bounded bowl-shaped domain, and u be a continuous real-valued function on D. The parabolic normal mapping of u at a point (x_0,t_0) is the set-valued function $P_u(x_0,t_0)$ given by

$$\{(p,h) : u(x,t) \geq u(x_0,t_0) + p\cdot(x-x_0) \text{ for all } t \leq t_0 \text{ and } x \in D(t),$$
$$h = p \cdot x_0 - u(x_0,t_0)\}.$$

As before, the parabolic normal mapping of a set $E \subset D$ is defined to be the union of the parabolic normal maps of each point in the set. The family of subsets E of D for which $P_u(E)$ is Lebesgue measurable is a Borel σ-algebra and the map $M_p(E) = |P_u(E)|$ is a measure, called the parabolic Monge-Ampère measure associated to the function u. These results are proved in [16]. We remark that, becuase of the translation invariance of the Lebesgue measure, the parabolic Monge-Ampère measure of a function u is identical to the parabolic Monge-Ampère measure of $u - \lambda$ for any constant λ.

We conclude this section with a parabolic analog of Aleksandrov's estimate (Theorem 1) due to Gutiérrez and Huang ([8]).

Theorem 3. *Let $D \subset \mathbb{R}^{n+1}$ be an open bounded bowl-shaped domain, and let $u \in C(\bar{D})$ be a parabolically convex function with $u = 0$ on $\partial_p D$. If $(x_0, t_0) \in D$, then*

$$|u(x_0, t_0)|^{n+1} \leq C_n \mathrm{dist}(x_0, \partial D(t_0)) \mathrm{diam}(D(t_0))^{n-1} M_p(D_{t_0})$$

where C_n is a dimensional constant, and M_p is the parabolic Monge-Ampère measure associated to u.

3. Parabolic estimates

In this section, we prove parabolic versions of Jerison's estimates. We adapt the arguments given in [10] to our situation. The first is the analog of Lemma 3.

Lemma 4. *Let D be a bounded, open bowl-shaped domain in $\mathbb{R}^{n+1}$. Suppose $u \in C(\bar{D})$ is parabolically convex and $u|_{\partial_p D} = 0$. Then there exists a dimensional constant C_n such that*

$$|u(x_0, t_0)|^{n+1} \leq C_n \delta(x_0, D(t_0)) |D(t_0)| |P_u(D_{t_0})|$$

for all $(x_0, t_0) \in D$, where $\delta(x_0, D(t_0))$ is the normalized distance from x_0 to the boundary of the n-dimensional convex set $D(t_0)$, and $|P_u(D_{t_0})| = M_p(D_{t_0})$ is the Lebesgue measure of the set $P_u(D_{t_0}) \subset \mathbb{R}^{n+1}$.

Proof $D(t_0)$ is a bounded convex subset of $\mathbb{R}^n$. By Lemma 1, we may choose an affine transformation T of $\mathbb{R}^n$ that normalizes $D(t_0)$. Define $\tilde{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $\tilde{T}(x,t) = (Tx, t)$. Then $\tilde{T}(D_{t_0}) \subset B_1(0) \times (-\infty, t_0]$. Let $v(z) = u(\tilde{T}^{-1} z)$ for $z \in \tilde{T}(D)$. Then $\tilde{T}(D)$ is a bowl-shaped domain, v is continuous on the closure of $\tilde{T}(D)$, is parabolically convex, and is zero on $\partial_p \tilde{T}(D)$.

Now apply the parabolic Aleksandrov estimate (Theorem 3) to v in $\tilde{T}(D)$ to obtain

$$|u(x_0, t_0)|^{n+1} = |v(\tilde{T}(x_0, t_0))|^{n+1}$$

$$\leq C_n \mathrm{dist}(Tx_0, \partial \tilde{T}(D(t_0))) [\mathrm{diam}(\tilde{T}(D(t_0)))]^{n-1} |P_v(\tilde{T}(D_{t_0}))|. \tag{5}$$

We next establish the following change-of-variable formula.

$$|P_v(\tilde{T}(D_{t_0}))| = |\det T^{-1}| \, |P_u(D_{t_0})|. \tag{6}$$

For simplicity, we make the following abuse of notation: when we write u or v as functions of x only, we mean the restrictions of u and v to $D(t_0)$. Let $p \in \partial u(x_0)$. Then

$$u(x, t_0) \geq u(x_0, t_0) + p \cdot (x - x_0)$$

for all $x \in D(t_0)$. Since u is non-increasing in t,

$$u(x, t) \geq u(x, t_0) \geq u(x_0, t_0) + p \cdot (x - x_0)$$

for all $t \leq t_0$ and $x \in D(t)$, so $(p, h) \in P_u(x_0, t_0)$ where $h = p \cdot x_0 - u(x_0, t_0)$. If $p \notin \partial u(x_0)$, then $(p, h) \notin P_u(x_0, t_0)$; therefore, $p \in \partial u(x_0)$ if and only if $(p, h) \in P_u(x_0, t_0)$. It is not hard to see that $p \in \partial u(x_0)$ if and only if $(T^{-1})^t p \in \partial v(Tx_0)$. Then as above, for $t \leq t_0$ and $y \in \tilde{T}(D)(t)$,

$$v(y, t) \geq v(y, t_0) + (T^{-1})^t p \cdot (y - Tx_0).$$

Hence, $(T^{-1})^t p \in \partial v(Tx_0)$ if and only if $((T^{-1})^t p, \tilde{h}) \in P_v(Tx_0, t_0)$, where $\tilde{h} = (T^{-1})^t p \cdot Tx_0 - v(Tx_0, t) = p \cdot x_0 - u(x_0) = h$. In other words, $(p, h) \in P_u(x_0, t_0)$ if and only if $((T^{-1})^t p, h) \in P_v(Tx_0, t_0)$. We also have $((T^{-1})^t p, h) = (\tilde{T}^{-1})^t (p, h)$ which implies that

$$(\tilde{T}^{-1})^t P_u(E) = P_v(\tilde{T}(E))$$

for any Borel set $E \subset D$. In particular, $(\tilde{T}^{-1})^t P_u(D_{t_0}) = P_v(\tilde{T}(D_{t_0}))$. This implies that

$$|\det \tilde{T}^{-1}| \, |P_u(D_{t_0})| = |P_v(\tilde{T}(D_{t_0}))|,$$

but $\det \tilde{T}^{-1} = \det T^{-1}$, showing (6).

Then using equation (6), Lemma 2, inequality (5), and the fact that $|\det T^{-1}| \leq C(n)|D(t_0)|_n$, we prove the claimed estimate:

$$\begin{aligned} |u(x_0, t_0)|^{n+1} &\leq C_n \delta(Tx_0, T(D(t_0)))|P_v(\tilde{T}(D_{t_0}))| \\ &= C_n \delta(x_0, D(t_0))|P_v(\tilde{T}(D_{t_0}))| \\ &= C_n \delta(x_0, D(t_0))|\det T^{-1}||P_u(D_{t_0})| \\ &\leq C_n \delta(x_0, D(t_0))|D(t_0)||P_u(D_{t_0})|. \end{aligned}$$

□

The next result extends Theorem 2 to the parabolic setting.

Theorem 4. *Let $0 < \epsilon \leq 1$. Let E be a bounded open bowl-shaped domain in $\mathbb{R}^{n+1}$, such that $E \subset B_1(0) \times (-\infty, \infty)$. Suppose $u \in C(\bar{E})$ is parabolically convex and zero on $\partial_p E$. Let M_p be the parabolic Monge-Ampère measure associated to u. Then there exists $C = C(\epsilon, n)$ such that*

$$|u(x_0, t_0)|^{n+1} \leq C \delta(x_0, E(t_0))^\epsilon \int_{E_{t_0}} \delta(x, E(t_0))^{1-\epsilon} \, dM_p(x, t).$$

for all $(x_0, t_0) \in E$.

Proof Without loss of generality, we may assume that $u(x_0, t_0) = -1$ (if this is not the case, multiply u by a suitably chosen positive constant). Let $s_k = s2^{-k\beta}$ where s and β are positive and chosen to satisfy $\beta(n+1) \leq \epsilon$ and $\sum_{k=1}^{\infty} s_k \leq \frac{1}{2}$. Let A denote the quantity

$$A := \delta(x_0, E(t_0))^{\epsilon} \int_{E_{t_0}} \delta(x, E(t_0))^{1-\epsilon} \, dM_p(x,t).$$

It suffices to show that $A \geq C(s)$, a constant depending on s and ϵ.

For $k = 1, 2, \ldots$, let $E_k = \{(x,t) \in E : u(x,t) \leq \lambda_k = -1 + s_1 + \cdots + s_k\}$. Define $E_0 = \{(x,t) \in E : u(x,t) \leq -1\}$. Notice that $E_k \subset E_{k+1}$ for $k = 1, 2, \ldots$, and that $E_0 \neq \emptyset$. Each of the sets E_k is bowl-shaped and $u|_{\partial_p E_k} = \lambda_k$ (taking $\lambda_0 = -1$). Fix t and let $\delta_k(t) = \operatorname{dist}(\partial E_k(t), \partial E(t))$.

Since $\delta_k(t) \not\to 0$ as $k \to \infty$ (if $\delta_k(t) \to 0$, then u would be smaller than $-\frac{1}{2}$ somewhere on $\partial_p E$), we may choose k to be the smallest nonnegative integer for which $\delta_{k+1}(t) > \frac{1}{2}\delta_k(t)$.

Let $x_k \in \partial E_k(t)$ be a point closest to $\partial E(t)$. Then we have that

$$\operatorname{dist}(x_k, \partial E_{k+1}(t)) < \frac{1}{2}\delta_k(t) < \delta_{k+1}(t). \tag{7}$$

The second of these inequalities holds because of the choice of k. The first inequality requires the following geometric argument. Let L be a line segment of length δ_k from x_k to $\partial E(t)$. The segment L meets $\partial E_{k+1}(t)$ at a point, x_{k+1}. Let ℓ represent the length of the part of L that connects $\partial E_{k+1}(t)$ to $\partial E(t)$. Then

$$\begin{aligned} \delta_k = |x_k - x_{k+1}| + \ell &\geq |x_k - x_{k+1}| + \operatorname{dist}(\partial E_{k+1}(t), \partial E(t)) \\ &= |x_k - x_{k+1}| + \delta_{k+1} \\ &> |x_k - x_{k+1}| + \frac{1}{2}\delta_k. \end{aligned}$$

Therefore, $\frac{1}{2}\delta_k > |x_k - x_{k+1}| \geq \operatorname{dist}(x_k, \partial E_{k+1}(t))$.

Now apply Lemma 4 to the function $u(x,t) - \lambda_{k+1}$ on the set E_{k+1} to get

$$|u(x_k, t) - \lambda_{k+1}|^{n+1} \leq C_n \delta(x_k, E_{k+1}(t)) |E_{k+1}(t)| \, M_p((E_{k+1})_t).$$

The point $x_k \in \partial E_k(t)$, so $u(x_k, t) = \lambda_k$ and $|u(x_k,t) - \lambda_{k+1}| = |\lambda_k - \lambda_{k+1}| = s_{k+1}$. Thus,

$$s_{k+1}^{n+1} \leq C_n \delta(x_k, E_{k+1}(t)) |E_{k+1}(t)| \, M_p((E_{k+1})_t). \tag{8}$$

Let L_t be a shortest segment from x_k to $\partial E_{k+1}(t)$ and let $z \in \partial E_{k+1}(t)$ be the other endpoint of L_t. Let ρ denote

$$\rho = |L_t| = |x_k - z| = \operatorname{dist}(x_k, \partial E_{k+1}(t)). \tag{9}$$

Since the set $E_{k+1}(t)$ is convex, the hyperplane Π (of dimension $n-1$) normal to L_t through z is a support plane for $E_{k+1}(t)$. Let Π' be the support plane parallel to Π on the opposite side of $E_{k+1}(t)$, so that $E_{k+1}(t)$ is contained between the two planes, and let $r = \operatorname{dist}(\Pi, \Pi')$. Then since $E_{k+1}(t) \subset B_1(0)$, there exists a constant $C = C(n)$ such that

$$|E_{k+1}(t)| \leq Cr. \tag{10}$$

We remark that the C in (10) can be chosen to be the volume of the unit ball in $\mathbb{R}^{n-1}$.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation normalizing $E_{k+1}(t)$. Then $\operatorname{dist}(T(\Pi), T(\Pi'))$ is bounded between two dimensional constants C_1 and C_2, with $C_1 < C_2$, and $C_1 \frac{\rho}{r} \leq \operatorname{dist}(Tx_k, T(\Pi)) \leq C_2 \frac{\rho}{r}$.

By Lemma 2, we have

$$\begin{aligned} \delta(x_k, E_{k+1}(t)) &= \delta(Tx_k, T(E_{k+1}(t))) \\ &\leq C \operatorname{dist}(Tx_k, \partial T(E_{k+1}(t))) \\ &\leq C \operatorname{dist}(Tx_k, T(\Pi)) \\ &\leq C \frac{\rho}{r}. \end{aligned}$$

Inserting this inequality into (8) and using (10), we get

$$\begin{aligned} s_{k+1}^{n+1} &\leq C \frac{\rho}{r} |E_{k+1}(t)| \, M_p((E_{k+1})_t) \leq C\rho \, M_p((E_{k+1})_t) \\ &< C\delta_{k+1}(t) \, M_p((E_{k+1})_t), \end{aligned}$$

where the last inequality holds since $\rho < \delta_{k+1}(t)$ (see (7) and (9)). Therefore we have

$$s_{k+1}^{n+1} < C\delta_{k+1}(t) \, M_p((E_{k+1})_t). \tag{11}$$

Since u is non-increasing in t and $E_0(t_0) \neq \emptyset$, $\delta_0(t)$ is defined for any $t \geq t_0$. On the other hand, for some values of t, $\delta_0(t)$ might not be defined; for instance, this is the case when $u > -1$ on $E(t)$. Then for any $t \geq t_0$, by the choice of k, we have $\delta_{k+1}(t) < \delta_k(t) \leq 2^{-k}\delta_0(t)$.

Since $\delta_0(t_0) \leq \operatorname{dist}(x_0, \partial E(t_0))$ and $\operatorname{diam}(E(t_0)) \leq 2$, we may conclude by Lemma 2*(c)* that $2^{-k}\delta_0(t_0) \leq C2^{-k}\delta(x_0, E(t_0))$ for a dimensional constant C.

Therefore,

$$\begin{aligned} \delta_{k+1}(t_0) \, M_p((E_{k+1})_{t_0}) &= \delta_{k+1}(t_0)^\epsilon \int_{(E_{k+1})_{t_0}} \delta_{k+1}(t_0)^{1-\epsilon} \, dM_p(y,s) \\ &\leq C2^{-k\epsilon}\delta(x_0, E(t_0))^\epsilon \int_{(E_{k+1})_{t_0}} \delta_{k+1}(t_0)^{1-\epsilon} \, dM_p(y,s) \\ &\leq C2^{-k\epsilon}\delta(x_0, E(t_0))^\epsilon \int_{(E_{k+1})_{t_0}} \delta(y, E(t_0))^{1-\epsilon} dM(y,s). \end{aligned} \tag{12}$$

The last inequality holds since

$$\begin{aligned} \delta_{k+1}(t_0) = \operatorname{dist}(\partial E_{k+1}(t_0), \partial E(t_0)) &\leq \operatorname{dist}(y, \partial E(t_0)) \\ &\leq C\delta(y, E(t_0)) \end{aligned}$$

for all $y \in E_{k+1}(t_0)$. Then from (11) and (12) we obtain that

$$\begin{aligned} s_{k+1}^{n+1} &\leq C2^{-k\epsilon}\delta(x_0, E(t_0))^{\epsilon} \int_{(E_{k+1})_{t_0}} \delta(y, E(t_0))^{1-\epsilon}\, dM_p(y,s) \\ &\leq C2^{-k\epsilon}A. \end{aligned}$$

Now recall that

$$s_{k+1}^{n+1} = s^{n+1}2^{-(n+1)(k+1)\beta} \geq s^{n+1}2^{-\epsilon(k+1)}$$

since $\beta(n+1) \leq \epsilon$. Hence

$$s^{n+1}2^{-\epsilon(k+1)} \leq C2^{-k\epsilon}A \Rightarrow s^{n+1} \leq CA,$$

where C depends on ϵ, so $A \geq C(s)$ as desired. □

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