MATH 302

Test 1 Solutions
October 16, 2007

Name ______________________

Calculators may be used. Put your answers in the space provided circle your answers. The maximum score on the test is 30 points.

1. 2 points Disprove the following statements:

1a. $(a, b) = ax + by$ for any integers x and y.

Many examples possible *e.g.* $(2, 3) = 1$ but $2 \cdot 5 + 3 \cdot 7 = 31 \neq 1$

1b. If a and b are relatively prime, then $(2a + b, a + 2b) = 3$.

Many examples possible *e.g.* $(2, 3) = 1$ but $(2 \cdot 2 + 3, 2 + 2 \cdot 3) = (7, 8) = 1 \neq 3$

2. 4 points Prove: If $a|b$ and $b|c$, then $a|c$.

Since $a|b$, there is a k so that $b = ka$. Likewise there is an n so that $c = nb$

Then $c = nb = nka$ and $a|c$

3. 4 points List the elements in a complete residue system modulo 24 which are not divisible by 2 or 3

$\{1, 5, 7, 11, 13, 17, 19, 23\}$ or $\{\pm 1, \pm 3, \pm 5, \pm 11\}$

Prove that if n is an odd integer not divisible by 3, then $n^2 \equiv 1 \pmod{24}$

n	n^2	$n^2 \pmod{24}$
± 1	1	1
± 5	25	1
± 7	49	1
± 11	121	1

4. 4 points Prove using congruences that $5|3^{3n+1} + 2^{n+1}$ for all $n > 0$.

$$3^{3n+1} + 2^{n+1} \equiv 27^n \cdot 3 + 2^n \cdot 2 \equiv 2^n \cdot (-2) + 2^n \cdot 2 \equiv 0 \pmod{5}$$

5. 4 points The greatest common divisor of 1500 and 840 is 60. Find three pairs of integers x and y for which $1500x + 840y = 60$.

It is equivalent to solve $25x + 14y = 1$

The Euclidean algorithm yields

$$\begin{aligned} 25 &= 1 \cdot 14 + 11 \\ 14 &= 1 \cdot 11 + 3 \\ 11 &= 3 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \end{aligned}$$

so $1 = 3 - 2 = 3 - (11 - 3 \cdot 3) = 4 \cdot 3 - 11 = 4 \cdot (14 - 11) - 11 = 4 \cdot 14 - 5 \cdot 11 = 4 \cdot 14 - 5(25 - 14) = 9 \cdot 14 - 5 \cdot 25$

5. Continued.

So $25 \cdot (-5) + 14 \cdot 9 = 1$ and one solution is $\boxed{x = -5,\ y = 9}$

Now $25 \cdot (-5 + 14t) + 14 \cdot (9 - 25t) = 1$ for any choice of integer t. Two such solutions would be $\boxed{x = 9,\ y = -16}$ and $\boxed{x = -19,\ y = 34}$

6. [4 points] Prove: If $(a, b) = 1$, and $(a, c) = 1$, then $(a, bc) = 1$.

$ax + by = 1$ and $at + cs = 1$ so $1 = (ax + by)(at + cs) = a^2xt + axcs + byat + bcys =$

$$= a(axt + cxs + byt) + bc(ts)$$

So $(a, bc) = 1$

7. [4 points] Find all least nonnegative incongruent solutions of the the congruence $\boxed{18x \equiv 12 \pmod{27}}$

Since $(18, 27) = 9 \not| 12$, There are no solutions.

8. [2 points] Find the inverse modulo $m = 99$ of $n = 48$.

We must solve $48x \equiv 1 \pmod{99}$. But $(48, 99) = 3 \not| 1$ So there is no inverse.

9. [2 points] Find seven consecutive composite numbers.

$\boxed{8! + 2,\ 8! + 3,\ 8! + 4,\ 8! + 5,\ 8! + 6,\ 8! + 7,\ 8! + 8}$