MATH 304

Quiz 1
January 19, 2007 Name ____________________

Put your answers in the space provided. Show your reasoning. Calculators may not be used. The maximum score on the quiz is 6 points. Fasten this sheet to your neatly written work.

1. 2 points Without using your calculator find the characteristic polynomial and the eigenvalues of the matrix $A = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 1 & 5 \\ 1 & 6 & 1 \end{bmatrix}$.

$$p(\lambda) = \begin{vmatrix} 7-\lambda & 1 & 0 \\ 0 & 1-\lambda & 5 \\ 1 & 6 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 7-\lambda & 1 & 0 \\ 0 & 1-\lambda & 5 \\ -6+\lambda & 5 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 7-\lambda & 1 & 0 \\ 0 & 1-\lambda & 5 \\ -6+\lambda & 6-\lambda & 6-\lambda \end{vmatrix} =$$

$$= (6-\lambda)\begin{vmatrix} 7-\lambda & 1 & 0 \\ 0 & 1-\lambda & 5 \\ -1 & 1 & 1 \end{vmatrix} = (-1)(6-\lambda)\begin{vmatrix} -7+\lambda & 8-\lambda & 7-\lambda \\ 0 & 1-\lambda & 5 \\ 1 & 0 & 0 \end{vmatrix} = (-1)(6-\lambda)\begin{vmatrix} 8-\lambda & 7-\lambda \\ 1-\lambda & 5 \end{vmatrix} =$$

$$= (-1)(6-\lambda)(40-5\lambda-7+8\lambda-\lambda^2) = \boxed{(6-\lambda)(\lambda^2-3\lambda-33)}$$

The eigenvalues are then $\boxed{6, \frac{3+\sqrt{141}}{2} = 7.43717, \frac{3-\sqrt{141}}{2} = -4.43717}$

2. 1 point Find x so that $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector for the matrix $A = \begin{bmatrix} 6 & x \\ 1 & -3 \end{bmatrix}$. Find the corresponding eigenvalue λ.

$$A\mathbf{u} = \begin{bmatrix} 6 & x \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6-2x \\ 7 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So to find the eigenvalue we solve $-2\lambda = 7$ so $\boxed{\lambda = -\frac{7}{2}}$

$-\frac{7}{2} = (6-2x)$. This yields $\boxed{x = \frac{19}{4}}$

3. Consider the following five vectors in $\mathbf{R}^4$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 3 \\ 6 \\ 3 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 2 \\ -2 \\ -4 \\ -2 \end{bmatrix},$$

They are the columns of the following matrix A and the matrix B is its reduced echelon form:

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & -1 & 3 & -2 \\ 1 & 3 & -2 & 2 & -4 \\ 1 & 3 & -1 & 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3a. [2 points] Find a basis for the null space of A.

From matrix B we have
$$\begin{aligned} x_1 &= -3x_2 \\ x_2 &= x_2 \\ x_3 &= -2x_5 \\ x_4 &= 0 \\ x_5 &= x_5 \end{aligned}$$
. Any vector in the null space of A is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Our basis for the null space of B is $\boxed{\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}}$

3b. [1 points] Find a basis for the column space of A.

From matrix B columns $1, 3, 4$ are pivot columns so a basis for the column space of A will be

$$\boxed{\left\{ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}}$$