MATH 304

Test 1 Solutions
January 30 , 2007

Name ____________________________

Put your answers in the space provided. Show your reasoning. The maximum score on the test is 30 points. Calculators may be used unless specifically restricted.

1. 6 points Diagonalize the matrix $A = \begin{bmatrix} 0.4 & -0.3 \\ 0.4 & 1.2 \end{bmatrix}$. Circle your answer. Use this diagonalization to show that A^k approaches $\begin{bmatrix} -0.5 & -0.75 \\ 1.0 & 1.50 \end{bmatrix}$ as $k \to \infty$.

$$\begin{vmatrix} 0.4-\lambda & -.03 \\ 0.4 & 1.2-\lambda \end{vmatrix} = 0.48 - 1.6\lambda + \lambda^2 + 0.12 = \lambda^2 - 1.6\lambda + 0.6 = (\lambda - 0.6)(\lambda - 1)$$

$\lambda = 1.$ $\begin{bmatrix} -0.6 & -0.3 \\ 0.4 & 0.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ so the eigenvector is $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 0.6.$ $\begin{bmatrix} -0.2 & -0.3 \\ 0.4 & 0.6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ so the eigenvector is $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Thus $A = \begin{bmatrix} 0.4 & -0.3 \\ 0.4 & 1.2 \end{bmatrix} = P \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix} P^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ -2 & -2 \end{bmatrix}$ and $P^{-1} = \frac{1}{4}\begin{bmatrix} -2 & -3 \\ 2 & 1 \end{bmatrix}$.

$A^k = PD^kP^{-1}$. $D^k = \begin{bmatrix} 1 & 0 \\ 0 & 0.6^k \end{bmatrix}$ and $0.6^k \to 0$ as $k \to \infty$. So $D^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Finally, $A^k \to P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 3 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 2 & 1 \end{bmatrix} = \boxed{\frac{1}{4}\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}}$

2. 4 points Suppose that $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ. Show that $\mathbf{x}$ is an eigenvector of $5I - A$. What is the corresponding eigenvalue?

$(5I - A)\mathbf{x} = 5\mathbf{x} - A\mathbf{x} = 5\mathbf{x} - \lambda\mathbf{x} = (5 - \lambda)\mathbf{x}$

So $\mathbf{x}$ is an eigenvector for $5I - A$ and the eigenvalue is $\boxed{5 - \lambda}$

3. 4 points For what value(s) of w is the matrix $\begin{bmatrix} 3 & 0 \\ 1 & w \end{bmatrix}$ diagonalizable?

If $w \neq 3$, then the eigenvalues are 3 and w; each with a linearly independent eigenvector and so A will be diagonalizable.

If $w = 3$ then 3 is an eigenvalue of algebraic multiplicity 2. We find its eigenvectors.

$A - 3I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. All eigenvectors will be multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We do not have enough linearly

independent eigenvectors with which to diagonalize A. So A is diagonalizable $\boxed{\text{for all } w \text{ except } 3}$.

4. 4 points Without using your calculator, find the eigenvalues for the matrix $A = \begin{bmatrix} 6 & -5 \\ 1 & 2 \end{bmatrix}$ and then find the corresponding eigenvectors.

$\begin{vmatrix} 6-\lambda & -5 \\ 1 & 2-\lambda \end{vmatrix} = 12 - 8\lambda + \lambda^2 + 5 = \lambda^2 - 8\lambda + 17$. So $\lambda = \dfrac{8 \pm \sqrt{64-68}}{2} = \boxed{4 \pm i}$.

$\lambda = 4 + i$. $\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2-i \\ 0 & 0 \end{bmatrix}$. The eigenvector is $\boxed{\begin{bmatrix} 2+i \\ 1 \end{bmatrix}}$

The eigenvector for $\boxed{\lambda = 4 - i \text{ is } \begin{bmatrix} 2-i \\ 1 \end{bmatrix}}$.

5. 5 points Find an invertible matrix P and a matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for which the matrix $A = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$ has the form PCP^{-1}. Encircle your answers.

$\begin{vmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1$ so $\lambda - 2 = \pm i$ or $\lambda = 2 \pm i$

$\lambda = 2 - i$. $\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}$. Eigenvector is $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

So $\boxed{C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}$ and $\boxed{P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}$

6. 2 points Without using your calculator, find the eigenvalue of the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ for the eigenvector $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

Since $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\boxed{\text{the eigenvalue is } 2}$.

7. 5 points Let $B = \begin{bmatrix} a & b & b & b & b \\ b & a & b & b & b \\ b & b & b & b & b \\ b & b & a & b & b \\ b & b & b & a & b \end{bmatrix}$.

7a. Explain why $a - b$ is an eigenvalue for B and give the dimension of the corresponding eigenspace.

$B-(a-b)I = \begin{bmatrix} b & b & b & b & b \\ b & b & b & b & b \\ b & b & -a+2b & b & b \\ b & b & a & -a+2b & b \\ b & b & b & a & -a+2b \end{bmatrix}$ is obviously singular. Thus $B-(a-b)I$ has a nontrivial null space and so $a-b$ is an eigenvalue for B.

If $b \neq 0, b \neq a$, then $B-(a-b)I \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ spans the eigenspace and so the eigenspace has dimension 1.

If $a = b \neq 0$ then the eigenvalues are 0 and $5b$. The eigenspace for $\lambda = 0$ has dimension 4 and the eigenspace for $\lambda = 5b$ has dimension 1. If $a = b = 0$, then the only eigenvalue is $\lambda = 0$ and everything is an eigenvector. Thus the dimension of the eigenspace is 5.

7b. Determine the other eigenvalue(s).

The matrix B was to be all b's except for a's along the diagonal. In cutting the problem down from a 6×6 matrix to a 5×5 I messed up. This part of the problem will be thrown out.