MATH 304

Test 3 Solutions
March 6, 2007

Name ____________________

Put your answers in the space provided. Show your reasoning. The maximum score on the test is 30 points.
Calculators may be used unless specifically restricted.

1. 2 points Let $\mathbf{w} = [6, -2, 3]^T$. Construct the matrix $I - 2\mathbf{w}\mathbf{w}^T$ Circle your answer.

$$I - 2\mathbf{w}\mathbf{w}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} [6,\ -2,\ 3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 36 & -12 & 18 \\ -12 & 4 & -6 \\ 18 & -6 & 9 \end{bmatrix} =$$

$$= \boxed{\begin{bmatrix} -71 & 24 & -36 \\ 24 & -7 & 12 \\ -36 & 12 & -17 \end{bmatrix}}$$

2. 6 points For any unit vector $\mathbf{u}$ in $\mathbf{R}^n$ let $Q = I - 2\mathbf{u}\mathbf{u}^T$.

2a. Is Q a symmetric matrix? Explain.

$$Q^T = I - 2\left(\mathbf{u}\mathbf{u}^T\right)^T = I - 2\mathbf{u}\mathbf{u}^T = Q.$$ So $Q = Q^T$ and is therefore symmetric.

2b. Show that $\mathbf{u}$ is an eigenvector of Q.

Recall that for the unit vector $\mathbf{u}$, $\mathbf{u}^T\mathbf{u} = 1$. Then $Q\mathbf{u} = \mathbf{u} - 2\mathbf{u}\mathbf{u}^T\mathbf{u} = \mathbf{u} - 2\mathbf{u} = -\mathbf{u}$.

Hence $\mathbf{u}$ is an eigenvector with eigenvalue -1

2c. Is Q idempotent, i.e. does $Q^2 = Q$? Explain.

$$Q^2 = \left(I - 2\mathbf{u}\mathbf{u}^T\right)\left(I - 2\mathbf{u}\mathbf{u}^T\right) = I - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T =$$

$$= I - 4\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T = I.$$ So Q is not idempotent.

3. 5 points Find a singular value decomposition, SVD, for the matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}$.

$$A^TA = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$ whose eigenvalues are 9 and 4

So $\boxed{V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}$ and $\boxed{\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}$

$$A\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{3}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \qquad A\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{2}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

By observation $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\boxed{U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & \sqrt{5} \\ -1 & 2 & 0 \end{bmatrix}}$ $\qquad A = U\Sigma V^T$

4. Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 6 \end{bmatrix}$

4a. 2 points Find the projection of $\mathbf{v}$ onto W. Circle your answer.

4b. 2 points Determine an orthogonal basis for $W^\perp$. Circle your answer twice,

$$\text{proj}_W \mathbf{v} = \frac{2+0+1+6}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2+0-1-6}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \frac{-2+0+1-6}{4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} =$$

$$= \frac{9}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} - \frac{7}{4}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \boxed{\frac{1}{4}\begin{bmatrix} 11 \\ -3 \\ 7 \\ 21 \end{bmatrix}}$$

The vector $\frac{1}{4}\begin{bmatrix} 8 \\ 0 \\ 4 \\ 24 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 11 \\ -3 \\ 7 \\ 21 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 \\ 3 \\ -3 \\ 3 \end{bmatrix}$ is a non-zero vector which spans $W^\perp$ and so an orthogonal basis for $W^\perp$ is $\boxed{\boxed{\left\{ \frac{1}{4}\begin{bmatrix} -3 \\ 3 \\ -3 \\ 3 \end{bmatrix} \right\}}}$ or $\boxed{\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}}}$

5. Let $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

5a. 3 points Orthogonally diagonalize A. Circle your answer.

The eigenvalues of A are 5 and -5, with corresponding eigenvectors $[1, 2]^T$ and $[-2, 1]^T$

$$\boxed{A = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}$$

5b. 3 points Find a spectral decomposition for A.

$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T =$

$$= \boxed{5\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} + (-5)\begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}} = \boxed{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}}$$

6. 4 points Find P and the new quadratic form when one makes the change of variable, $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $Q(\mathbf{x}) = x_1^2 - x_2^2 + 4x_1x_3 - 4x_2x_3$ into a quadratic form with no cross-product term.

$$Q(\mathbf{x}) = [x_1\ x_2\ x_3]\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A\mathbf{x}.$$ We need the eigenvalues and eigenvectors of A.

These eigenvalues are 3, 0 and -3 with corresponding eigenvectors $\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ and $\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$.

So $\mathbf{x} = P\mathbf{y}$ with $P = \frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ 1 & -2 & -2 \\ -2 & 1 & -2 \end{bmatrix}$. $\quad Q(\mathbf{y}) = \mathbf{y}PDP^T\mathbf{y}$

The quadratic form with no cross-product term is $Q(\mathbf{y}) = 3y_1^2 - 3y_3^2$.

7. 4 points Consider the quadratic form $Q(\mathbf{x}) = 3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_1x_3 + 4x_2x_3$. Find two orthogonal unit vectors at which $Q(\mathbf{x})$ is maximized subject to $\|\mathbf{x}\| = 1$. Hint: Eigenvalues of the quadratic form are 7 and -2.

$$Q(\mathbf{x}) = [x_1\ x_2\ x_3]\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A\mathbf{x}.$$ Subject to $\|\mathbf{x}\| = 1$, the maximum value of Q will be 7, the largest eigenvalue. This eigenvalue has geometric multiplicity 2, so we need to find two orthogonal eigenvectors for the eigenvalue 7.

$$A - 7I = \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = & x_1 & \\ x_2 = & -2x_2 & +2x_3 \\ x_3 = & & x_3 \end{matrix}$$

so $\mathbf{t}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\mathbf{t}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ are independent eigenvectors for A. Gram-Schmidt yields

$\mathbf{t}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\mathbf{t}_3 = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$ as orthogonal eigenvectors. So one answer is $\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\frac{1}{3\sqrt{5}}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

Alternatively an eigenvector for $\lambda = -2$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ from which we guess $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$ as

orthogonal eigenvectors for $\lambda = 7$. This gives the answer $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$