THE EILENBERG-WATTS THEOREM OVER SCHEMES

A. NYMAN

Abstract. We study obstructions to a direct limit preserving right exact functor F between categories of quasi-coherent sheaves on schemes being isomorphic to tensoring with a bimodule. When the domain scheme is affine, or if F is exact, all obstructions vanish and we recover the Eilenberg-Watts Theorem. This result is crucial to the proof that the noncommutative Hirzebruch surfaces constructed in [6] are noncommutative $\mathbb{P}^1$-bundles in the sense of [10].

Contents

1. Introduction

In this paper we describe a version of the Eilenberg-Watts Theorem over schemes. In order to motivate our results we first recall the Eilenberg-Watts Theorem proved independently by Eilenberg [3] and Watts [11]:

Theorem 1.1. *Let k be a commutative ring, let R and S be k-algebras and let $\mathsf{Mod}R$ (resp. $\mathsf{Mod}S$) denote the category of right R-modules (resp. right S-modules). If*

Date: January 2, 2010.

Key words and phrases. Eilenberg-Watts Theorem, Morita theory, non-commutative Hirzebruch surface.

2000 *Mathematics Subject Classification.* Primary 18F99; Secondary 14A22, 16D90, 18A25.

The author was partially supported by the National Security Agency under grant NSA H98230-05-1-0021.

$F : \mathsf{Mod}R \to \mathsf{Mod}S$ is a k-linear right exact functor commuting with direct limits, then there exists a k-central $R-S$-bimodule M such that $F \cong - \otimes_R M$.

The bimodule M in the previous theorem is easy to describe. $M = F(R)$ as a right-module, and its left-module structure is defined as follows: for each $r \in R$, we let $\phi_r \in \mathrm{Hom}_R(R, R)$ denote left multiplication by r. For $m \in M$, we define $r \cdot m := F(\phi_r)m$.

It is natural to ask if such a result holds when the categories $\mathsf{Mod}R$ and $\mathsf{Mod}S$ are replaced by categories of quasi-coherent sheaves on schemes X and Y, $\mathsf{Qcoh}X$ and $\mathsf{Qcoh}Y$. In order to precisely pose the question in this context, we need to introduce some notation. To this end, if Z is a scheme, X and Y are Z-schemes, $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_{X\times_Z Y}$-module, and the projections $X \times_Z Y \to X, Y$ are denoted pr_1 and pr_2, we define

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{E} := \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{M} \otimes_{\mathcal{O}_{X\times_Z Y}} \mathcal{E}).$$

We make the further assumption that

$$- \otimes_{\mathcal{O}_X} \mathcal{E} : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y,$$

which is automatic if $X \to Z$ is quasi-compact, separated and Z is affine.

Now let k be a commutative ring and let $Z = \operatorname{Spec} k$. Although the functor $- \otimes_{\mathcal{O}_X} \mathcal{E} : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ is not always right exact, it is *locally* right exact in the sense that if $u : U \to X$ is an open immersion from an affine scheme to X, then $u_*(-) \otimes_{\mathcal{O}_X} \mathcal{E} : \mathsf{Qcoh}U \to \mathsf{Qcoh}Y$ is right exact (see the proof that (2) is an isomorphism in Section 3). This suggests that a natural generalization of Theorem 1.1 to the case of functors between quasi-coherent sheaves on schemes would involve a characterization of locally right exact k-linear functors $F : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ commuting with direct limits. However, since (globally) right exact functors $F : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ appear naturally in the construction of certain non-commutative ruled surfaces (see the remark following Theorem 1.6 for more details), and since our motivation for studying generalizations of Theorem 1.1 comes from attempts to better understand these constructions, we specialize our study to right exact functors. It is thus natural for us to ask the following

Question 1.2. Let $F : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ denote a k-linear, right exact functor commuting with direct limits. Is F isomorphic to tensoring with a bimodule, i.e. does there exist an object $\mathcal{E}$ of $\mathsf{Qcoh}\mathcal{O}_{X\times_Z Y}$ such that $F \cong - \otimes_{\mathcal{O}_X} \mathcal{E}$?

When X is affine, we recall in Proposition 2.2 that the answer to this question is yes. Proposition 2.2 follows from a generalization of Theorem 1.1 proved in [7].

In general, the answer to this question is *no*, as the following example illustrates.

Example 1.3. [10, Example 3.1.3] Suppose k is a field, $X = \mathbb{P}^1_k$ and $Y = Z = \operatorname{Spec} k$. If $F = H^1(X, -)$, then F is k-linear, right exact, and commutes with direct limits. However, as we will prove in Proposition 5.4, F is not isomorphic to tensoring with a bimodule.

The purpose of this paper is to study the obstructions to a k-linear right exact functor $F : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ which commutes with direct limits being isomorphic to tensoring with a bimodule. In order to state our main result, we introduce notation and conventions which will be employed throughout the paper.

We let k denote a commutative ring, $Z = \mathrm{Spec}k$ and we assume all schemes and products of schemes are over Z. We assume X is a quasi-compact and separated scheme and Y is a separated scheme.

We note that the category

$$\mathsf{Funct}(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$$

of functors from $\mathsf{Qcoh}X$ to $\mathsf{Qcoh}Y$ is abelian, and we denote the full subcategory of k-linear functors (see Section 2 for a precise definition of *k-linear functor*) in $\mathsf{Funct}(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ by

$$\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y).$$

The category $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is abelian as well. We denote the full subcategory of $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ consisting of right exact functors commuting with direct limits by

$$\mathsf{Bimod}_k(X - Y).$$

We denote the full subcategory of $\mathsf{Bimod}_k(X-Y)$ consisting of functors which take coherent objects to coherent objects by

$$\mathsf{bimod}_k(X - Y).$$

The following definition, studied in Section 4, plays a central role in our theory.

Definition. An object F of $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is *totally global* if for every open immersion $u : U \to X$ with U affine, $Fu_* = 0$.

The functor F in Example 1.3 is totally global.

In order to generalize the Eilenberg-Watts Theorem, we first study an assignment, which we call the *Eilenberg-Watts functor*,

$$W : \mathsf{Bimod}_k(X - Y) \to \mathsf{Qcoh}X \times Y,$$

whose construction was sketched in [10, Lemma 3.1.1]. We prove that it is functorial (Subsection 5.2), left-exact (Proposition 5.1), compatible with affine localization (Proposition 5.2), and has the property that if $F \cong - \otimes_{\mathcal{O}_X} \mathcal{F}$ then $W(F) \cong \mathcal{F}$ (Proposition 5.4). It follows from Propostion 2.2 that if X is affine, then $F \cong - \otimes_{\mathcal{O}_X} W(F)$.

We then work towards our main result, established in Section 6:

Theorem 1.4. *If $F \in \mathsf{Bimod}_k(X - Y)$, then there exists a natural transformation*

$$\Gamma_F : F \longrightarrow - \otimes_{\mathcal{O}_X} W(F)$$

such that $\ker \Gamma_F$ *and* $\operatorname{cok} \Gamma_F$ *are totally global (Corollary 6.7). Furthermore, Γ_F is an isomorphism if*

1. *X is affine or*
2. *F is exact (Corollary 6.8) or*
3. *$F \cong - \otimes_{\mathcal{O}_X} \mathcal{F}$ for some object $\mathcal{F}$ in $\mathsf{Qcoh}X \times Y$ (Proposition 6.4).*

As a consequence, if $F \in \mathsf{Bimod}_k(X-Y)$, then $-\otimes_{\mathcal{O}_X} W(F)$ serves as the "best" approximation of F by tensoring with a bimodule in the following sense (Corollary 6.5):

Corollary. *Let $\mathcal{F}'$ be an object of $\mathsf{Qcoh}X \times Y$ and suppose $F' := - \otimes_{\mathcal{O}_X} \mathcal{F}'$ is an object in $\mathsf{Bimod}_k(X - Y)$. If $\Phi : F \to F'$ is a morphism in $\mathsf{Bimod}_k(X - Y)$, then Φ factors through Γ_F.*

In order to describe necessary and sufficient conditions for Γ_F to be an isomorphism, we introduce some notation: Let $\{U_i\}$ be a finite affine open cover of X, let $U_{ij} := U_i \cap U_j$, and let $u_i : U_i \to X$ and $u^i_{ij} : U_{ij} \to U_i$ denote inclusions. If $\mathcal{M}$ is an object of $\mathsf{Qcoh}X$, there is a canonical morphism (defined by (37))

$$\delta_{\mathcal{M}} : \oplus_i u_{i*}u_i^*\mathcal{M} \longrightarrow \oplus_{i<j} u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{M}$$

which is essentially the beginning of the sheafified Čech complex. We prove the following (Corollary 6.2):

Theorem 1.5. *If $F \in \mathsf{Bimod}_k(X - Y)$ then Γ_F is an isomorphism if and only if*

1. *for all flat objects $\mathcal{L}$ in $\mathsf{Qcoh}X$, the canonical map $F \ker \delta_{\mathcal{L}} \to \ker F\delta_{\mathcal{L}}$ is an isomorphism, and*
2. *$- \otimes_{\mathcal{O}_X} W(F)$ is right exact.*

The first item in Theorem 1.5 says that F must be close to being flat-acyclic, hence close to being a tensor product. The second item in Theorem 1.5 implies that $- \otimes_{\mathcal{O}_X} W(F)$ is in $\mathsf{Bimod}_k(X - Y)$.

Theorem 1.4 suggests that in order to obtain more precise information about objects in $\mathsf{Bimod}_k(X-Y)$ one must have a better understanding of the structure of totally global functors. While a general structure theory of totally global functors seems far off, we begin a very specialized investigation of this subject in Section 7. In particular, we classify totally global functors in $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ when k is an algebraically closed field. Our result in this direction is the following (Corollary 7.13):

Theorem 1.6. *If $F \in \mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ is totally global, then F is a direct sum of cohomologies, i.e. there exist integers $m, n_i \geq 0$ such that*

$$F \cong \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}.$$

We conclude the introduction by mentioning an application of Theorem 1.4(2). In [6], Ingalls and Patrick show that the blow-up of a noncommutative weighted projective space is a noncommutative Hirzebruch surface in an appropriate sense. More precisely, they show that the blow-up is a projectivization of an exact functor $F : \mathsf{Qcoh}\mathbb{P}^1 \to \mathsf{Qcoh}\mathbb{P}^1$ which commutes with direct limits. It follows from Theorem 1.4(2) that $F \cong - \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{F}$ where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$-module. This provides a crucial step in the proof that the noncommutative Hirzeburch surface Ingalls and Patrick construct is a noncommutative ruled surface in the sense of [10].

An apology for including proofs that diagrams commute: This paper contains a number of "technical" proofs that various diagrams commute. While some readers may frown upon the practice of including such proofs, we thought it wise to include them for the following reasons:

First, we are interested in proving a version of Theorem 1.4 in which Y is a non-commutative space (see [9, Section 1.2] for the definition of quasi-scheme, which is what we mean by non-commutative space). The proof of such a result will require the proof that diagrams similar to those in this paper commute. Since local arguments are often unavailable in the non-commutative setting, it will be important to have a careful record of which proofs of commutativity can be reduced to arguments global on Y (which should carry over without change to the non-commutative setting), and which are local on Y (which will have to be replaced by global arguments in the non-commutative setting).

Second, it is sometimes very difficult, even for extremely experienced mathematicians, to decide which diagrams commute for elementary reasons and which commute for deeper reasons. This fact is evidenced by the need for [2] to fill gaps in [5]. The gaps were not widely recognized as substantial until many years after the publication of [5]. Although the diagrams appearing in this paper are far less complicated than those studied in [2], we felt it important to save the skeptical reader from reconstructing the often tedious arguments on their own.

Acknowledgements: I am grateful to S. Paul Smith for numerous helpful conversations, for clarifying the proof of Proposition 2.2 and for allowing me to include some of his results in Section 7. I am also grateful to Daniel Chan for showing me how to generalize an earlier version of Theorem 1.4(2).

Finally, I thank Quan Shui Wu for hosting me at Fudan University during the 2006-2007 academic year, during which parts of this paper were written.

2. The Eilenberg-Watts Theorem

The purpose of this section is to recall the naive generalization of the Eilenberg-Watts Theorem that holds when the domain scheme is affine (see Proposition 2.2 for a precise formulation of this statement). The result is used implicitly in [10, Example 3.1.3]. We first recall the following definition, which is invoked in the statement of Proposition 2.2.

Definition 2.1. Recall that $Z = \operatorname{Spec} k$, let $f : X \to Z$ denote the k-scheme structure map for X and let $g : Y \to Z$ denote the k-scheme structure map for Y. An element $F \in \mathsf{Funct}(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is *k-linear* if the diagram

$$\begin{array}{ccc} k \times \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) & \to & k \times \operatorname{Hom}_{\mathcal{O}_Y}(F\mathcal{M}, F\mathcal{N}) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) & \to & \operatorname{Hom}_{\mathcal{O}_Y}(F\mathcal{M}, F\mathcal{N}) \end{array}$$

whose horizontal arrows are induced by F, and whose vertical arrows are induced by the k-module structure on $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ coming from global sections of the structure maps $\mathcal{O}_Z \to f_*\mathcal{O}_X$ and $\mathcal{O}_Z \to g_*\mathcal{O}_Y$ respectively, commutes.

Proposition 2.2. [7, Example 4.2] *If X is affine, then the inclusion functor*

$$\mathsf{Qcoh}(X \times Y) \to \mathsf{Bimod}_k(X - Y)$$

induced by the assignment $\mathcal{F} \mapsto - \otimes_{\mathcal{O}_X} \mathcal{F}$ is an equivalence of categories.

Proposition 2.2 follows from a general form of the Eilenberg-Watts Theorem [7, Theorem 3.1] characterizing right exact functors $F : \mathsf{Mod}R \to \mathsf{A}$ commuting with direct limits, where R is a ring, $\mathsf{Mod}R$ denotes the category of right R-modules and A is an abelian category.

We recall the proof that the inclusion functor in Proposition 2.2 is essentially surjective since we will invoke it in the sequel. We first construct an object, $\mathcal{F}$, of $\mathsf{Qcoh}(X \times Y)$ whose image is isomorphic to $F \in \mathsf{Bimod}_k(X - Y)$ as follows: Let $X = \operatorname{Spec} R$, and let $U \subset Y$ be affine open. We first define an $R \otimes_k \mathcal{O}_Y(U)$-module, N. We let N have underlying set and right-module structure equal to $F(\mathcal{O}_X)(U)$. We let

$$\mu_r \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

correspond to multiplication by $r \in R \cong \Gamma(X, \mathcal{O}_X)$, and we give N an R-module structure by defining $r \cdot n := F(\mu_r)(U)n$ for $n \in F(\mathcal{O}_X)(U)$. It remains to show that N is k-central, but this follows directly from the fact that F is k-linear. We conclude that N is an $R \otimes_k \mathcal{O}_Y(U)$-module, hence corresponds to a quasi-coherent $\mathcal{O}_{X\times U}$-module, $\mathcal{F}_U$. It is straightforward to check that the sheaves $\mathcal{F}_U$ glue to give a quasi-coherent $\mathcal{O}_{X\times Y}$-module, which we call $\mathcal{F}$.

We next construct an isomorphism $\Theta : - \otimes_{\mathcal{O}_X} \mathcal{F} \to F$ as follows: Let $\mathcal{M}$ be an $\mathcal{O}_X$-module and let $U \subset Y$ be an affine open subset. We define a morphism $\Theta_{\mathcal{M}}(U) : \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F}(U) \to F\mathcal{M}(U)$. To this end, we note that

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F}(U) &= \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{M} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F})(U) \\ &= (\mathrm{pr}_1^* \mathcal{M} \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F})(X \times U) \\ &\cong \mathcal{M}(X) \otimes_R F(\mathcal{O}_X)(U). \end{aligned}$$

Hence, in order to define $\Theta_{\mathcal{M}}(U)$, it suffices to construct an $\mathcal{O}_Y(U)$-module map $w : \mathcal{M}(X) \otimes_R F(\mathcal{O}_X)(U) \to F\mathcal{M}(U)$. This is constructed as in the proof of the Eilenberg-Watts Theorem, as follows. Suppose $m \in \mathcal{M}(X)$, $n \in F(\mathcal{O}_X)(U)$, $r \in R$, and

$$\mu_m \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M})$$

corresponds to the homomorphism in $\mathrm{Hom}_R(R, \mathcal{M}(X))$ sending 1 to m. Then

$$F(\mu_m) \in \mathrm{Hom}_{\mathcal{O}_Y}(F(\mathcal{O}_X), F\mathcal{M})$$

and

$$\begin{aligned} F(\mu_{mr})(U)(n) &= F(\mu_m\mu_r)(U)(n) \\ &= F(\mu_m)(U)F(\mu_r)(U)(n) \\ &= F(\mu_m)(U)(rn). \end{aligned}$$

Hence, the function $w(m \otimes n) := F(\mu_m)(U)(n)$ extends to a well defined homomorphism of $\mathcal{O}_Y(U)$-modules $w : \mathcal{M}(X) \otimes_R F(\mathcal{O}_X)(U) \to F\mathcal{M}(U)$, which in turn corresponds to a map of $\mathcal{O}_Y(U)$-modules $\Theta_{\mathcal{M}}(U) : \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F}(U) \to F\mathcal{M}(U)$. It is straightforward to show that the maps $\Theta_{\mathcal{M}}(U)$ glue to give a map of $\mathcal{O}_Y$-modules

$$\Theta_{\mathcal{M}} : \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F} \to F\mathcal{M}$$

and that $\Theta_{\mathcal{M}}$ is an isomorphism which is natural in $\mathcal{M}$.

In the sequel, we will often refer to Θ as the canonical isomorphism in the proof of Proposition 2.2.

3. Basechange, the Projection Formula, and Compatibilities

Our construction of the Eilenberg-Watts functor and our proof of Theorem 1.4 depends, in a fundamental way, on the existence and properties of two canonical isomorphisms which are constructed using basechange and the projection formula. The purpose of this section is to describe these isomorphisms as well as several fundamental compatibilities involving them.

Throughout this section, we let U denote an affine scheme, we let $u : U \to X$ denote an open immersion, we let $v = u \times \mathrm{id}_Y$, and we let $p, q : U \times Y \to U, Y$ denote projections.

We begin with some preliminary observations. We note that the diagram

$$\begin{array}{ccc} U \times Y & \xrightarrow{p} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ X \times Y & \xrightarrow[\mathrm{pr}_1]{} & X \end{array}$$

is a fiber square. We claim the basechange and projection formula morphisms

$$\mathrm{pr}_1^*\, u_* \longrightarrow v_* v^*\, \mathrm{pr}_1^*\, u_* \xrightarrow{\cong} v_* p^* u^* u_* \longrightarrow v_* p^*$$

and

$$v_* - \otimes_{\mathcal{O}_{X\times Y}} - \to v_* v^* (v_* - \otimes_{\mathcal{O}_{X\times Y}} -) \xrightarrow{\cong} v_* (v^* v_* - \otimes_{\mathcal{O}_{U\times Y}} v^* -) \to v_* (- \otimes_{\mathcal{O}_{U\times Y}} v^* -)$$

induced by unit and counit morphisms of (u^*, u_*) and (v^*, v_*), and by the distributivity of pullbacks over tensor products, are isomorphisms. To this end, we note that it suffices to prove that they are isomorphisms over subsets of the form $V \times W$ where $V \subset X$ and $W \subset Y$ are open affine subsets. This reduces the claim to a straightforward affine computation, which we omit.

Let $\mathcal{E} \in \mathsf{Qcoh} U \times Y$ and $\mathcal{F} \in \mathsf{Qcoh} X \times Y$. We define canonical isomorphisms

$$u^*(-) \otimes_{\mathcal{O}_U} \mathcal{E} \longrightarrow - \otimes_{\mathcal{O}_X} v_* \mathcal{E} \tag{1}$$

and

$$u_*(-) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow - \otimes_{\mathcal{O}_U} v^* \mathcal{F}, \tag{2}$$

natural in $\mathcal{E}$ and $\mathcal{F}$, as follows: The map (1) is defined to be the composition

$$\begin{aligned} u^*(-) \otimes_{\mathcal{O}_U} \mathcal{E} &\xrightarrow{=} q_*(p^* u^* - \otimes_{\mathcal{O}_{U\times Y}} \mathcal{E}) \\ &\xrightarrow{\cong} q_*(v^*\, \mathrm{pr}_1^* - \otimes_{\mathcal{O}_{U\times Y}} \mathcal{E}) \\ &\xrightarrow{=} \mathrm{pr}_{2*}\, v_* (v^*\, \mathrm{pr}_1^* - \otimes_{\mathcal{O}_{U\times Y}} \mathcal{E}) \\ &\xrightarrow{\cong} \mathrm{pr}_{2*} (\mathrm{pr}_1^* - \otimes_{\mathcal{O}_{X\times Y}} v_* \mathcal{E}) \\ &\xrightarrow{=} - \otimes_{\mathcal{O}_X} v_* \mathcal{E} \end{aligned}$$

where the second morphism comes from the equality $\mathrm{pr}_1\, v = up$ and the fourth morphism is the projection formula.

We define the map (2) as the composition

$$\begin{aligned} u_*(-) \otimes_{\mathcal{O}_X} \mathcal{F} &\xrightarrow{=} \mathrm{pr}_{2*} (\mathrm{pr}_1^*\, u_* - \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}) \\ &\xrightarrow{\cong} \mathrm{pr}_{2*} (v_* p^* - \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F}) \\ &\xrightarrow{\cong} \mathrm{pr}_{2*}\, v_* (p^* - \otimes_{\mathcal{O}_{U\times Y}} v^* \mathcal{F}) \\ &\xrightarrow{=} q_* (p^* - \otimes_{\mathcal{O}_{U\times Y}} v^* \mathcal{F}) \end{aligned}$$

where the second morphism is basechange and the third morphism is the projection formula.

Naturality of (1) and (2) follows from naturality of basechange and the projection formula.

The remainder of this section is devoted to the proof that (1) and (2) satisfy three compatibilities. The first says that (1) and (2) are compatible with the units and counits of the adjoint pairs (u^*, u_*) and (v^*, v_*) (Lemma 3.1). The second says that if $\widetilde{U} \xrightarrow{\tilde{u}} U$ is an open affine immersion and $\tilde{v} = \tilde{u} \times \mathrm{id}_Y$, then (1) and (2)

are compatible with the canonical isomorphisms $(u\tilde{u})^* \cong \tilde{u}^*u^*$ and $(v\tilde{v})^* \cong \tilde{v}^*v^*$ (Lemma 3.2). The third says that (1) and (2) are compatible with affine basechange (Lemma 3.3).

Lemma 3.1. *Consider the following diagram*

$$\begin{array}{ccc} -\otimes_{\mathcal{O}_X}\mathcal{F} & \longrightarrow & -\otimes_{\mathcal{O}_X} v_*v^*\mathcal{F} \\ \downarrow & & \downarrow \\ u_*u^*(-)\otimes_{\mathcal{O}_X}\mathcal{F} & \longrightarrow & u^*(-)\otimes_{\mathcal{O}_U} v^*\mathcal{F} \end{array} \tag{3}$$

whose top horizontal and left vertical are induced by unit morphisms, whose right vertical is the inverse of (1) and whose bottom horizontal is induced by (2). Then this diagram commutes.

Similarly, consider the following diagram

$$\begin{array}{ccc} -\otimes_{\mathcal{O}_U}\mathcal{E} & \longleftarrow & -\otimes_{\mathcal{O}_U} v^*v_*\mathcal{E} \\ \uparrow & & \uparrow \\ u^*u_*(-)\otimes_{\mathcal{O}_U}\mathcal{E} & \longleftarrow & u_*(-)\otimes_{\mathcal{O}_X} v_*\mathcal{E} \end{array} \tag{4}$$

whose top horizontal and left vertical are induced by counit morphisms, whose bottom horizontal is induced by the inverse of (1), and whose right vertical is (2). Then this diagram commutes.

Proof. We first show that (3) commutes. Consider the following diagram

$$\begin{array}{ccc} \mathrm{pr}_1^*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & \longrightarrow & \mathrm{pr}_1^*(-)\otimes_{\mathcal{O}_{X\times Y}} v_*v^*\mathcal{F} \\ \downarrow & & \downarrow \\ \mathrm{pr}_1^*\, u_*u^*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & & v_*(v^*\,\mathrm{pr}_1^*(-)\otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ \downarrow & & \downarrow \\ v_*p^*u^*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & \longrightarrow & v_*(p^*u^*(-)\otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \end{array} \tag{5}$$

whose top horizontal and upper-left vertical are induced by unit morphisms, whose upper-right vertical and bottom horizontal are induced by the projection formula, whose bottom-left vertical is basechange and whose bottom-right vertical is canonical. It suffices to show that this diagram commutes.

To this end, we consider the following diagram

$$\begin{array}{ccc} \mathrm{pr}_1^* & \longrightarrow & v_*v^*\,\mathrm{pr}_1^* \\ \downarrow & & \downarrow \\ \mathrm{pr}_1^*\, u_*u^* & & v_*p^*u^* \\ \downarrow & & \downarrow \\ v_*v^*\,\mathrm{pr}_1^*\, u_*u^* & \longrightarrow & v_*p^*u^*u_*u^* \end{array} \tag{6}$$

whose top horizontal, left verticals and bottom-right vertical are induced by unit morphisms, and whose bottom horizontal and upper-right vertical are canonical.

We claim that this diagram commutes. The claim follows by splitting (6) into two subdiagrams via the morphism

$$v_*v^* \operatorname{pr}_1^* \longrightarrow v_*v^* \operatorname{pr}_1^* u_*u^*$$

induced by the unit of (u^*, u_*), and noticing that each commutes by the naturality of the unit.

Next consider the following diagram

$$\begin{array}{ccc} \operatorname{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} & \longrightarrow & \operatorname{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} v_*v^*\mathcal{F} \\ \downarrow & & \downarrow \\ v_*v^* \operatorname{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} & \longrightarrow & v_*(v^* \operatorname{pr}_1^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ \downarrow & & \downarrow \\ v_*p^*u^*(-) \otimes_{\mathcal{O}_{X\times Y}} \mathcal{F} & \longrightarrow & v_*(p^*u^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \end{array} \tag{7}$$

whose top horizontal and top-left vertical are induced by unit morphisms, whose top-right vertical and middle and bottom horizontals are induced by the projection formula, and whose bottom verticals are canonical. The claim implies that, in order to show (5) commutes, it suffices to show that both squares of (7) commute. The bottom square of (7) commutes by the naturality of the projection formula.

We next prove that the top square of (7) commutes. To this end, consider the following diagram

$$\begin{array}{ccc} - \otimes_{\mathcal{O}_{X\times Y}} - & \longrightarrow & - \otimes_{\mathcal{O}_{X\times Y}} v_*v^*(-) \\ \downarrow & & \downarrow \\ v_*v^*(-) \otimes_{\mathcal{O}_{X\times Y}} - & \longrightarrow & v_*(v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \end{array} \tag{8}$$

whose right vertical and bottom horizontal are projection formulas, and whose left vertical and top horizontal are induced by units. In order to prove that the top square of (7) commutes, it suffices to show that (8) commutes. To prove this, we note that the bottom route of (8) equals the bottom route in the diagram

$$\begin{array}{ccccc} - \otimes_{\mathcal{O}_{X\times Y}} - & \longrightarrow & v_*v^*(- \otimes_{\mathcal{O}_{X\times Y}} -) & \longrightarrow & v_*(v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \\ \downarrow & & \downarrow & & \uparrow \\ v_*v^* - \otimes_{\mathcal{O}_{X\times Y}} - & \longrightarrow & v_*v^*(v_*v^* - \otimes_{\mathcal{O}_{X\times Y}} -) & \longrightarrow & v_*(v^*v_*v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \end{array} \tag{9}$$

whose left arrows are unit morphisms, whose right horizontals are induced by distributivity of pullbacks over tensor products

$$v^*(- \otimes_{\mathcal{O}_{X\times Y}} -) \xrightarrow{\cong} v^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*(-) \tag{10}$$

and whose right horizontal is the counit morphism. We claim that (9) commutes. It will follow from the claim that the bottom route of (8) equals the composite of the top horizontals in (9). Similarly, the top route of (8) equals the composite of the top horizontals in (9). Therefore, the commutativity of (8), and hence the commutativity of (3) will follow from the commutativity of (9). We establish this now.

The left square of (9) commutes by naturality of the unit morphism. To prove the right square of (9) commutes, we consider the following diagram

$$\begin{array}{ccc} v_*v^*(-\otimes_{\mathcal{O}_{X\times Y}}-) & \longrightarrow & v_*(v^*(-)\otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \\ \downarrow & & \downarrow \\ v_*v^*(v_*v^*-\otimes_{\mathcal{O}_{X\times Y}}-) & \longrightarrow & v_*(v^*v_*v^*(-)\otimes_{\mathcal{O}_{U\times Y}} v^*(-)) \end{array}$$

whose verticals are induced by units and whose horizontals are (10). This diagram commutes by naturality of (10). It follows from this that the right square of (9) commutes as well.

The proof that (4) commutes is similar to the proof that (3) commutes, and we omit it. □

Lemma 3.2. *Suppose $\widetilde{U} \subset U$ are open affine subschemes of X, with inclusion morphisms $\tilde{u} : \widetilde{U} \to U$ and $u : U \to X$. Let $\tilde{v} = \tilde{u} \times \mathrm{id}_Y$, let $v = u \times \mathrm{id}_Y$, and let $\mathcal{F}$ be an object of $\mathsf{Qcoh}X \times Y$. Consider the following diagram*

$$\begin{array}{ccc} -\otimes_{\mathcal{O}_{\widetilde{U}}} \tilde{v}^*v^*\mathcal{F} & \longrightarrow & \tilde{u}_*(-)\otimes_{\mathcal{O}_U} v^*\mathcal{F} \\ \downarrow & & \downarrow \\ -\otimes_{\mathcal{O}_{\widetilde{U}}} (v\tilde{v})^*\mathcal{F} & \longrightarrow & (u\tilde{u})_*(-)\otimes_{\mathcal{O}_X}\mathcal{F} \end{array} \tag{11}$$

whose horizontals and right vertical are induced by the inverse of (2) and whose left vertical is induced by the canonical isomorphism $\tilde{v}^*v^* \xrightarrow{\cong} (v\tilde{v})^*$. Then this diagram commutes.

Similarly, consider the following diagram

$$\begin{array}{ccc} -\otimes_{\mathcal{O}_X} v_*\tilde{v}_*\mathcal{F} & \longrightarrow & u^*(-)\otimes_{\mathcal{O}_U}\tilde{v}_*\mathcal{F} \\ \downarrow & & \downarrow \\ (u\tilde{u})^*(-)\otimes_{\mathcal{O}_{\widetilde{U}}}\mathcal{F} & \xrightarrow{\cong} & \tilde{u}^*u^*(-)\otimes_{\mathcal{O}_{\widetilde{U}}}\mathcal{F} \end{array} \tag{12}$$

whose top horizontal and verticals are induced by the inverse of (1), and whose bottom horizontal is induced by the canonical isomorphism $(u\tilde{u})^* \cong \tilde{u}^*u^*$. Then this diagram commutes.

Proof. Let $p, q : U \times Y \to U, Y$ and $\tilde{p}, \tilde{q} : \widetilde{U} \times Y \to \widetilde{U}, Y$ denote projections. Consider the diagram

(13)

$$\begin{array}{ccccc} \mathrm{pr}_1^*(u\tilde{u})_*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & \longrightarrow & v_*p^*\tilde{u}_*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & \longrightarrow & v_*(p^*\tilde{u}_*(-)\otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ (v\tilde{v})_*\tilde{p}^*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & \xrightarrow{=} & v_*\tilde{v}_*\tilde{p}^*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & \longrightarrow & v_*(\tilde{v}_*\tilde{p}^*(-)\otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ \downarrow & & & & \downarrow \\ (v\tilde{v})_*(\tilde{p}^*(-)\otimes_{\mathcal{O}_{\widetilde{U}\times Y}}(v\tilde{v})^*\mathcal{F}) & & \longrightarrow & & v_*\tilde{v}_*(\tilde{p}^*(-)\otimes_{\mathcal{O}_{\widetilde{U}\times Y}}\tilde{v}^*v^*\mathcal{F}) \end{array}$$

whose top-left horizontal and top verticals are induced by basechange, whose top-right horizontal, middle-right horizontal and bottom verticals are induced by the projection formula, and whose bottom isomorphism is induced by the canonical

isomorphism $(v\tilde{v})^* \xrightarrow{\cong} \tilde{v}^* v^*$. Since $q = \mathrm{pr}_2\, v$ and $\tilde{q} = \mathrm{pr}_2\, v\tilde{v}$, in order to prove (11) commutes, it suffices to show that (13) commutes.

The upper-right square of (13) commutes by the naturality of the projection formula. The fact that the upper-left square of (13) commutes follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{pr}_1^*(u\tilde{u})_* & \xrightarrow{=} & \mathrm{pr}_1^*\, u_*\tilde{u}_* \\ \cong\downarrow & & \downarrow\cong \\ (v\tilde{v})_*\tilde{p}^* & \xrightarrow[\cong]{} & v_*p^*\tilde{u}_* \end{array} \tag{14}$$

whose non-trivial isomorphisms are induced by basechange. The commutativity of (14) can be checked affine locally and we omit the routine verification.

The commutativity of the bottom rectangle of (13) follows from the commutativity of the diagram

$$\begin{array}{ccc} (v\tilde{v})_*(-)\otimes_{\mathcal{O}_{X\times Y}}\mathcal{F} & \xrightarrow{\cong} & v_*(\tilde{v}_*(-)\otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \\ \cong\downarrow & & \downarrow\cong \\ (v\tilde{v})_*((-)\otimes_{\mathcal{O}_{\tilde{U}\times Y}}(v\tilde{v})^*\mathcal{F}) & \xrightarrow[\cong]{} & v_*\tilde{v}_*((-)\otimes_{\mathcal{O}_{\tilde{U}\times Y}}\tilde{v}^*v^*\mathcal{F}) \end{array} \tag{15}$$

whose bottom horizontal is induced by the canonical isomorphism $(v\tilde{v})^* \xrightarrow{\cong} \tilde{v}^*v^*$ and whose other arrows are induced by the projection formula. The commutativity of (15) again follows from a routine affine computation, which we omit.

The proof that (12) commutes is similar and may be reduced to the commutativity of a diagram of the form (15) as well. We leave the details to the reader. □

Lemma 3.3. *Let $U_1, U_2 \subset X$ be affine open subschemes, let $U_{12} := U_1 \cap U_2$ with inclusions*

$$\begin{array}{ccc} U_{12} & \xrightarrow{u_{12}^1} & U_1 \\ {\scriptstyle u_{12}^2}\downarrow & & \downarrow{\scriptstyle u_1} \\ U_2 & \xrightarrow[u_2]{} & X. \end{array}$$

For $i = 1, 2$, let $v_i = u_i \times \mathrm{id}_Y$ and let $v_{12}^i = u_{12}^i \times \mathrm{id}_Y$. Let $\mathcal{E}$ be an object of $\mathsf{Qcoh}U_1 \times Y$, and consider the diagram

$$\begin{array}{ccc} u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_1}}\mathcal{E} & \longrightarrow & u_{12*}^1u_{12}^{2*}(-)\otimes_{\mathcal{O}_{U_1}}\mathcal{E} \\ \uparrow & & \downarrow \\ u_{2*}(-)\otimes_{\mathcal{O}_X} v_{1*}\mathcal{E} & & u_{12}^{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} v_{12}^{1*}\mathcal{E} \\ \uparrow & & \downarrow \\ -\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}\mathcal{E} & \longrightarrow & -\otimes_{\mathcal{O}_{U_2}} v_{12*}^2v_{12}^{1*}\mathcal{E} \end{array}$$

whose horizontals are induced by basechange, and whose verticals are induced by (1) and (2). Then this diagram commutes.

Proof. Consider the following diagram

(16)
$$\begin{array}{ccccc}
u_{2*}(-)\otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}v^{1*}_{12}\mathcal{E} & \longrightarrow & u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_1}} v^1_{12*}v^{1*}_{12}\mathcal{E} & \longrightarrow & u^{1*}_{12}u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} v^{1*}_{12}\mathcal{E} \\
\uparrow & & & & \downarrow \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}v^{1*}_{12}\mathcal{E} & & & & u^{2*}_{12}u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} v^{1*}_{12}\mathcal{E} \\
\uparrow & & & & \downarrow \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}\mathcal{E} & \longrightarrow & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}\mathcal{E} & \longleftarrow & u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}\mathcal{E}
\end{array}$$

whose top horizontals and bottom-right vertical are induced by (1), whose bottom-left horizontal is induced by basechange, whose bottom-right horizontal is induced by a counit, whose bottom-left vertical is induced by a unit, whose top-left vertical is induced by the inverse of (2), and whose top-right vertical is canonical. By the naturality of units, counits and the morphisms (1) and (2), and by the commutativity of (3), it suffices to prove that (16) commutes. To this end, we consider the diagram

$$\begin{array}{ccccc}
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}v^{1*}_{12}\mathcal{E} & \longleftarrow & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}v^1_{12*}v^{1*}_{12}\mathcal{E} \\
\uparrow & & \uparrow \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}\mathcal{E} & \longleftarrow & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}\mathcal{E}
\end{array}$$

whose verticals are induced by units and whose horizontals are induced by basechange. By the naturality of basechange, this diagram commutes. Hence, to prove that (16) commutes, it suffices to prove that if $\mathcal{F} := v^{1*}_{12}\mathcal{E}$, then the following diagram

(17)
$$\begin{array}{ccccc}
u_{2*}(-)\otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}\mathcal{F} & \longrightarrow & u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_1}} v^1_{12*}\mathcal{F} & \longrightarrow & u^{1*}_{12}u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} \mathcal{F} \\
\uparrow & & & & \downarrow \\
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}\mathcal{F} & & & & u^{2*}_{12}u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}} \mathcal{F} \\
\uparrow & & & & \downarrow \\
-\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}v^1_{12*}\mathcal{F} & \longrightarrow & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} & \longleftarrow & u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F}
\end{array}$$

whose bottom-left vertical is induced by basechange, whose bottom-left horizontal is induced by a counit, and whose other maps are identical to the maps in (16), commutes.

We complete the proof by showing that the diagram (17) commutes. To this end, we note that (17) can be broken into the following four subdiagrams: the diagram

(18)
$$\begin{array}{ccc}
-\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}\mathcal{F} & \xrightarrow{=} & -\otimes_{\mathcal{O}_{U_2}} v_2^*v_{2*}v^2_{12*}\mathcal{F} \\
\downarrow & & \downarrow \\
-\otimes_{\mathcal{O}_{U_2}} v^2_{12*}v^{1*}_{12}v^1_{12*}\mathcal{F} & \longrightarrow & -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F}
\end{array}$$

whose left vertical is induced by basechange and whose right vertical and bottom horizontal are counits, the diagram

$$\begin{array}{ccccc} u_{2*}(-)\otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}\mathcal{F} & \longrightarrow & u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_1}} v^1_{12*}\mathcal{F} & \longrightarrow & u_{12}^{1*}u_1^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}}\mathcal{F} \\ = \downarrow & & & & \downarrow \cong \\ u_{2*}(-)\otimes_{\mathcal{O}_X} v_{2*}v^2_{12*}\mathcal{F} & \longrightarrow & u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} & \longrightarrow & u_{12}^{2*}u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_{12}}}\mathcal{F} \end{array} \tag{19}$$

whose horizontals are induced by the inverse of (1) and whose right vertical is canonical, the diagram

$$\begin{array}{ccc} -\otimes_{\mathcal{O}_{U_2}} v_2^*v_{2*}v^2_{12*}\mathcal{F} & \longleftarrow & u_{2*}(-)\otimes_{\mathcal{O}_X} v_{2*}v^2_{12*}\mathcal{F} \\ \downarrow & & \downarrow \\ -\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} & \longleftarrow & u_2^*u_{2*}(-)\otimes_{\mathcal{O}_{U_2}} v^2_{12*}\mathcal{F} \end{array} \tag{20}$$

whose top horizontal is (2), whose right vertical is (1) and whose other arrows are counits, and the diagram

$$\begin{array}{ccc} u_{2*}(-)\otimes_{\mathcal{O}_X} v_{1*}v^1_{12*}\mathcal{F} & \xrightarrow{=} & u_{2*}(-)\otimes_{\mathcal{O}_X} v_{2*}v^2_{12*}\mathcal{F} \\ \uparrow & & \uparrow \\ -\otimes_{\mathcal{O}_{U_2}} v_2^*v_{1*}v^1_{12*}\mathcal{F} & \xrightarrow{=} & -\otimes_{\mathcal{O}_{U_2}} v_2^*v_{2*}v^2_{12*}\mathcal{F} \end{array} \tag{21}$$

whose verticals are induced by the inverse of (2). It suffices to show that these subdiagrams commute. The fact that diagram (18) commutes is left as an exercise to the reader. The fact that diagram (19) commutes follows from Lemma 3.2. The fact that diagram (20) commutes follows from Lemma 3.1, and the commutativity of (21) is trivial. □

4. Totally Global Functors

Our goal in this section is to define and study elementary properties of *totally global functors*.

Definition 4.1. We say $F \in \mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is *totally global* if for any open immersion $u : U \longrightarrow X$ with U affine, $Fu_* = 0$.

We note that this definition makes sense. For, $u : U \to X$ is an affine morphism since X is separated [4, II, ex. 4.3], so that u is quasi-compact and separated [4, II, ex. 5.17b]. Hence, u_* takes quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_X$-modules [4, II, Prop. 5.8c].

The following lemma explains the motivation behind the use of the term *totally global*.

Lemma 4.2. *Suppose X is noetherian. If F is totally global and $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module whose support lies in an affine open subset U of X (included via u), then $F\mathcal{M} = 0$.*

Proof. Since F commutes with direct limits and X is noetherian, it suffices to prove that $F\mathcal{M} = 0$ for $\mathcal{M}$ coherent. Let $i : \operatorname{Supp}\mathcal{M} \to X$ and $i' : \operatorname{Supp}\mathcal{M} \to U$ denote

inclusions, so that $i = ui'$. Since i is a closed immersion, the unit map $\mathcal{M} \to i_*i^*\mathcal{M}$ is an isomorphism. Thus,

$$\begin{aligned} F\mathcal{M} &\cong Fi_*i^*\mathcal{M} \\ &= F(ui')_*i^*\mathcal{M} \\ &= Fu_*{i'}_*i^*\mathcal{M} \\ &= 0. \end{aligned}$$

□

Example 4.3. Let W be a noetherian scheme. Then, for $i > 0$ the functor $H^i(W, -)$ is totally global by [4, III, ex. 8.2].

Proposition 4.4. *If $\mathcal{F}$ is an object of $\mathsf{Qcoh}X \times Y$ and $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$ is totally global then $\mathcal{F} = 0$.*

Proof. Suppose U is an affine scheme, $u : U \to X$ is an open immersion, $v = u \times \mathrm{id}_Y$ and $p, q : U \times Y \to U, Y$ are projections. The map (2) induces an isomorphism,

$$\begin{aligned} u_*(-) \otimes_{\mathcal{O}_X} \mathcal{F} &\xrightarrow{\cong} - \otimes_{\mathcal{O}_U} v^*\mathcal{F} \\ &\xrightarrow{=} q_*(p^*(-) \otimes_{\mathcal{O}_{U\times Y}} v^*\mathcal{F}) \end{aligned}$$

Since F is totally global,

$$\begin{aligned} 0 &= Fu_*\mathcal{O}_U \\ &\cong q_*v^*\mathcal{F} \\ &= \mathrm{pr}_{2*}(v_*v^*\mathcal{F}). \end{aligned}$$

Thus, if W is an affine open subset of Y, $v^*\mathcal{F}(U \times W) = 0$. Therefore, $v^*\mathcal{F} = 0$ since its sections on an affine open cover are 0. We conclude that if $p \in U \times Y$, then $\mathcal{F}_p = 0$. Since U is arbitrary, $\mathcal{F} = 0$ as desired. □

For the remainder of this section, we take *affine open cover of X* to mean a set of pairs $\{(U_i, u_i)\}$ where $u_i : U_i \to X$ is inclusion of an affine open subset U_i of X such that every point of X is contained in some U_i.

Proposition 4.5. *If $F \in \mathsf{Bimod}_k(X - Y)$ and $\{(U_i, u_i)\}$ is an affine open cover of X such that $Fu_{i*} = 0$ for all i, then F is totally global.*

Proof. We first prove that if X is affine, $\{(W_i, w_i)\}$ is an affine open cover of X, and $E \in \mathsf{Funct}_k(X - Y)$ is such that $Ew_{i*} = 0$ for all i, then $E = 0$. Since X is affine, $E \cong - \otimes_{\mathcal{O}_X} \mathcal{E}$ for some object $\mathcal{E}$ of $\mathsf{Qcoh}X \times Y$ by Proposition 2.2. Thus, if $p, q : W_i \times Y \to W_i, Y$ are projections and $v_i = w_i \times \mathrm{id}_Y$, then by (2),

$$Ew_{i*} \cong q_*(p^* - \otimes_{\mathcal{O}_{W_i\times Y}} {v_i}^*\mathcal{E}).$$

The vanishing of Ew_{i*} for all i implies that $q_*{v_i}^*\mathcal{E} = 0$ for all i. Therefore, for all i and all $W \subset Y$ open affine,

$${v_i}^*\mathcal{E}(W_i \times W) = 0.$$

This implies that ${v_i}^*\mathcal{E} = 0$ for all i which implies that $\mathcal{E}$, and hence E, is 0.

Now we prove that for any X and any affine open cover $\{(V_j, v_j)\}$ of X, $Fv_{j*} = 0$ for all j. The proposition will follow. Let F and $\{(U_i, u_i)\}$ be as in the statement of the Proposition. Let $w_{ij} : U_i \cap V_j \to V_j$ and $w'_{ij} : U_i \cap V_j \to U_i$ denote inclusions. Then $Fv_{j*}w_{ij*} = Fu_{i*}w'_{ij*} = 0$ for all i by hypothesis. But V_j is

affine, $Fv_{j*} \in \mathsf{Bimod}_k(V_j \times Y)$ since v_{j*} is right exact by the affineness of v_j [4, III, Prop. 8.1 and Remark 3.5.1], and $\mathfrak{W} := \{(U_i \cap V_j, w_{ij})\}_i$ is an affine open cover of V_j. Hence the argument of the first paragraph applies to the functor $E = Fv_{j*}$ and the open cover $\mathfrak{W}$ of V_j, so that $Fv_{j*} = 0$. □

5. The Eilenberg-Watts Functor

In this section we review the construction of an assignment

$$W : \mathsf{Bimod}_k(X - Y) \to \mathsf{Qcoh}X \times Y$$

sketched in [10, Lemma 3.1.1], and prove it is functorial (Subsection 5.2), left-exact (Proposition 5.1), and compatible with affine localization (Proposition 5.2). We will show in Corollary 6.5 that if $F \in \mathsf{Bimod}_k(X - Y)$ then $- \otimes_{\mathcal{O}_X} W(F)$ serves as a "best" approximation to F by tensoring with a bimodule. In order to prove this, we will need the fact, proven in Proposition 5.4, that if $F \cong - \otimes_{\mathcal{O}_X} \mathcal{F}$ then $W(F) \cong \mathcal{F}$. We end the section by showing that if F is exact, then $\mathrm{pr}_1^*(-) \otimes_{\mathcal{O}_{X\times Y}} W(F)$ is exact (Corollary 5.5). This result is used in Section 6 to prove that if F is exact then $F \cong - \otimes_{\mathcal{O}_X} W(F)$ (Corollary 6.8).

5.1. **Preliminaries.** Before defining the functor W, we describe conventions we will employ throughout the rest of this paper.

Let $\{U_i\}_{i\in I}$ be a collection of open subschemes of X (we identify the underlying set of U_i with a subset of the underlying set of X). For any finite subset $\{i_1, \ldots, i_n\}$ of I, we let

$$U_{i_1\cdots i_n} = U_{i_1} \cap \cdots \cap U_{i_n}$$

and we let

$$u_{i_1\cdots i_n} : U_{i_1\cdots i_n} \to X$$

denote the inclusion morphism. For any inclusion $\{j_1, \ldots, j_m\} \subset \{i_1, \ldots, i_n\}$ of finite subsets of I, we let

$$u^{j_1\cdots j_m}_{i_1\cdots i_n} : U_{i_1\cdots i_n} \to U_{j_1\cdots j_m}$$

denote the inclusion morphism. Similar conventions apply when the open subschemes are labelled $\{V_j\}$ or $\{W_k\}$, etc. We denote the open cover $\{U_i\}_{i\in I}$ by $\mathfrak{U}$.

For i, j with $j \neq i$, we let

$$\eta^i_{ij} : \mathrm{id}_{\mathsf{Qcoh}U_i} \to u^i_{ij*}u^{i*}_{ij}$$

denote the canonical unit of the adjoint pair (u^{i*}_{ij}, u^i_{ij*}).

5.2. **Definition of the Eilenberg-Watts Functor.** Let F be an object in the category $\mathsf{Bimod}_k(X-Y)$. Our goal in this subsection is to associate to F an object $W(F) \in \mathsf{Qcoh}X \times Y$, and show the assignment $F \mapsto W(F)$ is functorial. To this end, we first choose a finite affine open cover of X, $\mathfrak{U} = \{U_i\}_{i\in I}$ with $I = \{1, \ldots, n\}$. Recall that X is quasi-compact, so such an open cover exists.

For each $i \in I$, the proof of Proposition 2.2 gives us an $\mathcal{F}_i \in \mathsf{Qcoh}U_i \times Y$ and a canonical isomorphism $Fu_{i*} \xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i$.

Now let $V_i = U_i \times Y$. Recalling our notational conventions about open covers of $X \times Y$ in Section 5.1, we claim that there exists a canonical isomorphism

$$\psi_{ij} : v^{i*}_{ij}\mathcal{F}_i \xrightarrow{\cong} v^{j*}_{ij}\mathcal{F}_j. \tag{22}$$

To prove the claim, we note that there are isomorphisms

$$\begin{aligned} - \otimes_{\mathcal{O}_{U_{ij}}} v_{ij}^{i*}\mathcal{F}_i &\xrightarrow{\cong} u_{ij*}^i(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ &\xrightarrow{\cong} Fu_{i*}u_{ij*}^i \\ &\xrightarrow{=} Fu_{j*}u_{ij*}^j \\ &\xrightarrow{\cong} u_{ij*}^j(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \\ &\xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_{ij}}} v_{ij}^{j*}\mathcal{F}_j, \end{aligned}$$

the first is the inverse of (2), the second is from the definition of $\mathcal{F}_i$ and the fourth and fifth are defined similarly. The map ψ_{ij} corresponds to the composition above under the equivalence from Proposition 2.2.

Next, for each pair $i, j \in I$ with $j > i$, we let

$$\phi_i^{ij} : v_{i*}\mathcal{F}_i \longrightarrow v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i$$

denote the morphism induced by η_{ij}^i and we define

$$\phi_j^{ij} : v_{j*}\mathcal{F}_j \longrightarrow v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i$$

as the composition of the morphism $v_{j*}\mathcal{F}_j \longrightarrow v_{j*}v_{ij*}^j v_{ij}^{j*}\mathcal{F}_j = v_{i*}v_{ij*}^i v_{ij}^{j*}\mathcal{F}_j$ induced by η_{ij}^j and the morphism $v_{i*}v_{ij*}^i v_{ij}^{j*}\mathcal{F}_j \longrightarrow v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i$ induced by ψ_{ij}^{-1}.

Finally, since I is finite, in order to specify a morphism

$$\oplus_i v_{i*}\mathcal{F}_i \longrightarrow \oplus_{i<j} v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i,$$

it suffices to define a morphism

$$\theta_i^{jk} : v_{i*}\mathcal{F}_i \to v_{j*}v_{jk*}^j v_{jk}^{j*}\mathcal{F}_j$$

for all $i, j, k \in I$ with $j < k$. We define such a morphism as

$$\theta_i^{jk} = \begin{cases} \phi_i^{ik} & \text{if } i = j, \\ -\phi_i^{ji} & \text{if } i = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

The morphisms $\{\theta_i^{jk}\}$ induce a morphism

$$\theta_F : \oplus_i v_{i*}\mathcal{F}_i \longrightarrow \oplus_{i<j} v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i. \tag{24}$$

We define

$$W_{\mathfrak{U}}(F) := \ker \theta_F.$$

We next note that $W_{\mathfrak{U}}(F)$ is an object of $\mathsf{Qcoh}X \times Y$. For, since $v_{i*}v_{ij*}^i = v_{ij*}$ is an affine morphism, it is quasi-compact and separated by [4, II, ex. 5.17b]. Hence if $\mathcal{M}$ is an object of $\mathsf{Qcoh}U_{ij} \times Y$ then $v_{ij*}\mathcal{M}$ is an object of $\mathsf{Qcoh}X \times Y$ by [4, II, Prop. 5.8c].

We define $W_{\mathfrak{U}}$ on morphisms as follows. Let $\Delta : E \longrightarrow F$ be a morphism in $\mathsf{Bimod}_k(X - Y)$ and let $\Delta * u_{i*} : Eu_{i*} \longrightarrow Fu_{i*}$ denote the horizontal composition of the natural transformations Δ and $\mathrm{id}_{u_{i*}}$. By the proof of Proposition 2.2, there are canonical isomorphisms $Eu_{i*} \longrightarrow - \otimes_{\mathcal{O}_{U_i}} \mathcal{E}_i$ and $Fu_{i*} \longrightarrow - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i$. Hence, $\Delta * u_{i*}$ induces, via these isomorphisms, a morphism

$$- \otimes_{\mathcal{O}_{U_i}} \mathcal{E}_i \longrightarrow - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i.$$

Therefore, by Proposition 2.2, there is an induced morphism

$$\delta_i : \mathcal{E}_i \longrightarrow \mathcal{F}_i.$$

The fact that the maps $\{\delta_i\}_{i\in I}$ induce a morphism $\delta : W_{\mathfrak{U}}(E) \longrightarrow W_{\mathfrak{U}}(F)$ now follows from the naturality of η^i_{ij} and of ψ_{ij}. We leave it as a straightforward exercise for the reader to check that the naturality of ψ_{ij} follows from the naturality of (1) and (2).

We define

$$W_{\mathfrak{U}}(\Delta) := \delta.$$

It is straightforward to complete the verification that $W_{\mathfrak{U}}$ is a functor and we omit it.

5.3. **Properties of the Eilenberg-Watts Functor.** The following result will not be used in the sequel.

Proposition 5.1. *The functor $W_{\mathfrak{U}} : \mathsf{Bimod}_k(X - Y) \to \mathsf{Qcoh}X \times Y$ is left-exact in the sense that if $F', F, F'' \in \mathsf{Bimod}_k(X - Y)$ are such that*

$$0 \to F' \overset{\Lambda}{\to} F \overset{\Xi}{\to} F'' \to 0 \tag{25}$$

is exact in $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$, then

$$0 \to W_{\mathfrak{U}}(F') \overset{W_{\mathfrak{U}}(\Lambda)}{\to} W_{\mathfrak{U}}(F) \overset{W_{\mathfrak{U}}(\Xi)}{\to} W_{\mathfrak{U}}(F'')$$

is exact in $\mathsf{Qcoh}X \times Y$.

Proof. Exactness of (25) implies that, for all u_i,

$$0 \to F'u_{i*} \overset{\Lambda}{\to} Fu_{i*} \overset{\Xi}{\to} F''u_{i*} \to 0$$

is exact in $\mathsf{Funct}_k(U_i - Y)$. Thus, this sequence is exact in $\mathsf{Bimod}_k(U_i - Y)$. By Proposition 2.2, the induced sequence

$$0 \to \mathcal{F}'_i \to \mathcal{F}_i \to \mathcal{F}''_i \to 0$$

is exact in $\mathsf{Qcoh}U_i \times Y$. Therefore the induced sequences

$$0 \to \oplus_i v_{i*}\mathcal{F}'_i \to \oplus_i v_{i*}\mathcal{F}_i \to \oplus_i v_{i*}\mathcal{F}''_i \to 0$$

and

$$0 \to \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}'_i \to \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \to \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}''_i \to 0$$

are exact since v_i and v^i_{ij} are affine and v^i_{ij} is an open immersion. There is thus a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0\to & & \oplus_i v_{i*}\mathcal{F}'_i & \to & \oplus_i v_{i*}\mathcal{F}_i & \to & \oplus_i v_{i*}\mathcal{F}''_i & \to 0 \\
 & & \downarrow{\scriptstyle \theta_{F'}} & & \downarrow{\scriptstyle \theta_F} & & \downarrow{\scriptstyle \theta_{F''}} & \\
0\to & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}'_i & \to & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i & \to & \oplus_{i<j} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}''_i & \to 0 & \\
\end{array}$$

Left-exactness of $W_{\mathfrak{U}}$ follows from the Snake Lemma. □

Proposition 5.2. *The functor $W_{\mathfrak{U}}$ is compatible with affine localization in the sense that if $\mathfrak{U} \cap U_k$ denotes the affine open cover $\{U_{ik}\}_{i\in I}$ of U_k, then*

$$W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \cong v_k^* W_{\mathfrak{U}}(F)$$

naturally in F.

Proof. We prove the result in several steps.

Step 1: We note that the canonical basechange morphisms

$$v_k^* v_{i*} \longrightarrow v_{ik*}^k v_{ik}^{i*}$$

and

$$v_{ik}^{i*} v_{ij*}^i \longrightarrow v_{ijk*}^{ik} v_{ijk}^{ij*}$$

associated to the diagrams

$$\begin{array}{ccc} U_{ik} \times Y & \xrightarrow{v_{ik}^i} & U_i \times Y \\ {\scriptstyle v_{ik}^k}\downarrow & & \downarrow{\scriptstyle v_i} \\ U_k \times Y & \underset{v_k}{\longrightarrow} & X \times Y \end{array}$$

and

$$\begin{array}{ccc} U_{ijk} \times Y & \xrightarrow{v_{ijk}^{ij}} & U_{ij} \times Y \\ {\scriptstyle v_{ijk}^{ik}}\downarrow & & \downarrow{\scriptstyle v_{ij}^i} \\ U_{ik} \times Y & \underset{v_{ik}^i}{\longrightarrow} & U_i \times Y \end{array}$$

are isomorphisms. This follows from a routine affine computation, which we omit.
Step 2: Consider the composition

$$v_{ik*}^k v_{ik}^{i*} \to v_{ik*}^k v_{ik}^{i*} v_{ij*}^i v_{ij}^{i*} \to v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ij*} v_{ij}^{i*} \to v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ik*} v_{ik}^{i*}$$

whose left arrow is induced by the unit of the adjoint pair (v_{ij}^{i*}, v_{ij*}^i), whose middle arrow is induced by the second basechange isomorphism from Step 1, and whose right arrow is induced from the canonical isomorphism

$$v_{ijk}^{ij*} v_{ij}^{i*} \stackrel{\cong}{\to} (v_{ij}^i v_{ijk}^{ij})^* = (v_{ik}^i v_{ijk}^{ik})^* \stackrel{\cong}{\to} v_{ijk}^{ik*} v_{ik}^{i*}. \tag{26}$$

We note that this composition is equal to the morphism induced by the unit of the adjoint pair $(v_{ijk}^{ik*}, v_{ijk*}^{ik})$. In order to prove this fact, consider the following diagram

$$\begin{array}{ccc} v_{ik}^{i*} & \to & v_{ik}^{i*} v_{ij*}^i v_{ij}^{i*} \\ \downarrow & & \downarrow \\ v_{ijk*}^{ik} v_{ijk}^{ik*} v_{ik}^{i*} & \to & v_{ijk*}^{ik} v_{ijk}^{ij*} v_{ij}^{i*} \end{array}$$

whose top horizontal and left vertical are induced by canonical units, whose right vertical is induced by basechange isomorphisms from Step 1, and whose bottom horizontal is induced by the inverse of (26). It suffices to prove that this diagram commutes. The verification of this fact follows from a routine affine computation, which we omit.
Step 3: Let F be an object of $\mathsf{Bimod}_k(X - Y)$ and consider the morphism δ

$$\delta : \oplus_i v_{ik*}^k v_{ik}^{i*} \mathcal{F}_i \longrightarrow \oplus_{i<j} v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ik*} v_{ik}^{i*} \mathcal{F}_i$$

defined by the composition

$$\begin{aligned}
\oplus_i v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i &\xrightarrow{\cong} \oplus_i v^*_k v_{i*}\mathcal{F}_i \\
&\xrightarrow{v^*_k\theta_F} \oplus_{i<j} v^*_k v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\
&\xrightarrow{\cong} \oplus_{i<j} v^k_{ik*}v^{i*}_{ik}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\
&\xrightarrow{\cong} \oplus_{i<j} v^k_{ik*}v^{ik}_{ijk*}v^{ij*}_{ijk}v^{i*}_{ij}\mathcal{F}_i \\
&\xrightarrow{\cong} \oplus_{i<j} v^k_{ik*}v^{ik}_{ijk*}v^{ik*}_{ijk}v^{i*}_{ik}\mathcal{F}_i
\end{aligned}$$

whose first and third and fourth arrows are basechange isomorphisms from Step 1, and whose fifth arrow is induced by the canonical isomorphism (26). Let δ^{jl}_i denote the component of δ from the ith summand to the jlth summand, i.e.

$$\delta^{jl}_i : v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i \longrightarrow v^k_{jk*}v^{jk}_{jlk*}v^{jk*}_{jlk}v^{j*}_{jk}\mathcal{F}_j.$$

We show that

- $\delta^{jl}_i = 0$ *if* i *is not equal to* j *or* l,
- δ^{ij}_i *is induced by the unit of the adjoint pair* $(v^{ik*}_{ijk}, v^{ik}_{ijk*})$, *and*
- δ^{ij}_j *is equal to* -1 *times the composition*

$$v^k_{jk*}v^{j*}_{jk}\mathcal{F}_j \longrightarrow v^k_{jk*}v^{jk}_{ijk*}v^{jk*}_{ijk}v^{j*}_{jk}\mathcal{F}_j \longrightarrow v^k_{ik*}v^{ik}_{ijk*}v^{ik*}_{ijk}v^{i*}_{ik}\mathcal{F}_i$$

whose left arrow is induced by the unit of the adjoint pair $(v^{jk*}_{ijk}, v^{jk}_{ijk*})$ *and whose right arrow corresponds, under the equivalence of Proposition 2.2, to the composition of functors*

$$\begin{aligned}
-\otimes_{\mathcal{O}_{U_{ijk}}} v^{jk*}_{ijk}v^{j*}_{jk}\mathcal{F}_j &\xrightarrow{\cong} u^{jk}_{ijk*}(-)\otimes_{\mathcal{O}_{U_{jk}}} v^{j*}_{jk}\mathcal{F}_j \\
&\xrightarrow{\cong} u^j_{jk*}u^{jk}_{ijk*}(-)\otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \\
&\xrightarrow{\cong} Fu_{j*}u^j_{jk*}u^{jk}_{ijk*} \\
&\xrightarrow{=} Fu_{i*}u^i_{ik*}u^{ik}_{ijk*} \\
&\xrightarrow{\cong} -\otimes_{\mathcal{O}_{U_{ijk}}} v^{ik*}_{ijk}v^{i*}_{ik}\mathcal{F}_i
\end{aligned} \tag{27}$$

whose first two arrows are induced by the inverse of (2), whose third arrow is the canonical isomorphism from the proof of Proposition 2.2, and whose last arrow is defined analogously to the composition of the first three arrows.

The fact that $\delta^{jl}_i = 0$ if i is not equal to j or l follows from the definition of θ_F. The assertion regarding δ^{ij}_i follows from Step 2.

It remains to verify the description of δ_j^{ij}. Consider the following diagram

$$
\begin{array}{ccc}
v_k^* v_{j*} v_{ij*}^j v_{ij}^{j*} \mathcal{F}_j & \longrightarrow & v_k^* v_{i*} v_{ij*}^i v_{ij}^{i*} \mathcal{F}_i \\
\downarrow & & \downarrow \\
v_{jk*}^k v_{jk}^{j*} v_{ij*}^j v_{ij}^{j*} \mathcal{F}_j & & v_{ik*}^k v_{ik}^{i*} v_{ij*}^i v_{ij}^{i*} \mathcal{F}_i \\
\downarrow & & \downarrow \\
v_{jk*}^k v_{ijk*}^{jk} v_{ijk}^{ij*} v_{ij}^{j*} \mathcal{F}_j & & v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ij*} v_{ij}^{i*} \mathcal{F}_i \\
\downarrow & & \downarrow \\
v_{jk*}^k v_{ijk*}^{jk} v_{ijk}^{jk*} v_{jk}^{j*} \mathcal{F}_j & \longrightarrow & v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ik*} v_{ik}^{i*} \mathcal{F}_i
\end{array}
\tag{28}
$$

whose top horizontal is induced by the map ψ_{ji} defined in (22), whose bottom horizontal is induced by the morphism corresponding to (27) and whose verticals are induced by basechange isomorphisms from Step 1 and by canonical morphisms of the form (26). By Step 2, it suffices to prove that this diagram commutes. To this end, consider the diagrams

$$
\begin{array}{ccc}
v_k^* v_{j*} v_{ij*}^j & \stackrel{=}{\longrightarrow} & v_k^* v_{i*} v_{ij*}^i \\
\downarrow & & \downarrow \\
v_{jk*}^k v_{jk}^{j*} v_{ij*}^j & & v_{ik*}^k v_{ik}^{i*} v_{ij*}^i \\
\downarrow & & \downarrow \\
v_{jk*}^k v_{ijk*}^{jk} v_{ijk}^{ij*} & \underset{=}{\longrightarrow} & v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ij*}
\end{array}
\tag{29}
$$

whose verticals are induced by basechange morphisms, and the diagram

$$
\begin{array}{ccc}
v_{ijk}^{ij*} v_{ij}^{j*} \mathcal{F}_j & \stackrel{v_{ijk}^{ij*} \psi_{ji}}{\longrightarrow} & v_{ijk}^{ij*} v_{ij}^{i*} \mathcal{F}_i \\
\downarrow & & \downarrow \\
v_{ijk}^{jk*} v_{jk}^{j*} \mathcal{F}_j & \longrightarrow & v_{ijk}^{ik*} v_{ik}^{i*} \mathcal{F}_i
\end{array}
\tag{30}
$$

whose verticals are induced by canonical morphisms of the form (26) and whose bottom horizontal is the morphism corresponding to (27). In order to prove that (28) commutes, it suffices to prove that (29) and (30) commute.

The commutativity of (29) follows from a straightforward affine computation, which we omit. To prove that (30) commutes, we first note that $v_{ijk}^{ij*} \psi_{ji}$ corresponds to the composition (27) by the naturality of (2). Hence, a straightforward computation shows that the commutativity of (30) follows from the commutativity of four "corner" subdiagrams. The upper-left such diagram, for example, is the diagram

$$
\begin{array}{ccccc}
- \otimes_{\mathcal{O}_{U_{ijk}}} v_{ijk}^{ij*} v_{ij}^{j*} \mathcal{F}_j & \longrightarrow u_{ijk*}^{ij}(-) \otimes_{\mathcal{O}_{U_{ij}}} v_{ij}^{j*} \mathcal{F}_j & \longrightarrow & u_{ij*}^{j} u_{ijk*}^{ij}(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \\
\downarrow & & & \downarrow = \\
- \otimes_{\mathcal{O}_{U_{ijk}}} (v_{ij}^{j} v_{ijk}^{ij})^* \mathcal{F}_j & & \longrightarrow & (u_{ij}^{j} u_{ijk}^{ij})_*(-) \otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j
\end{array}
$$

whose horizontals are induced by the isomorphisms (2) and whose left vertical is induced by the canonical isomorphism

$$v_{ijk}^{ij*}v_{ij}^{j*} \xrightarrow{\cong} (v_{ij}^{j}v_{ijk}^{ij})^*.$$

These corner subdiagrams commute by Lemma 3.2.
Step 4: Let $\mathcal{E}_i \in \mathsf{Qcoh}U_{ik}\times Y$ *denote the object corresponding to the functor* $Fu_{ik*} \in \mathsf{Bimod}_k(U_{ik} - Y)$ *in the proof of Proposition 2.2. Consider the following diagram*

$$\begin{array}{ccc} v_{ijk}^{jk*}v_{jk}^{j*}\mathcal{F}_j & \longrightarrow & v_{ijk}^{ik*}v_{ik}^{i*}\mathcal{F}_i \\ \downarrow & & \downarrow \\ v_{ijk}^{jk*}\mathcal{E}_j & \longrightarrow & v_{ijk}^{ik*}\mathcal{E}_i \end{array} \tag{31}$$

whose top horizontal is the map (27), whose bottom horizontal is the map ψ_{ji} *defined by (22) but corresponding to the functor* Fu_{k*}*, whose left vertical is induced by the composition*

$$\begin{aligned} - \otimes_{\mathcal{O}_{U_{ijk}}} v_{ijk}^{jk*}v_{jk}^{j*}\mathcal{F}_j &\xrightarrow{\cong} u_{ijk*}^{jk}(-)\otimes_{\mathcal{O}_{U_{jk}}} v_{jk}^{j*}\mathcal{F}_j \\ &\xrightarrow{\cong} u_{jk*}^{j}u_{ijk*}^{jk}(-)\otimes_{\mathcal{O}_{U_j}} \mathcal{F}_j \\ &\xrightarrow{\cong} Fu_{j*}u_{jk*}^{j}u_{ijk*}^{jk} \\ &\xrightarrow{=} Fu_{jk*}u_{ijk*}^{jk} \\ &\xrightarrow{\cong} u_{ijk*}^{jk}(-)\otimes_{\mathcal{O}_{U_{jk}}} \mathcal{E}_j \\ &\xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_{ijk}}} v_{ijk}^{jk*}\mathcal{E}_j \end{aligned}$$

whose first, second, and sixth morphisms are induced by (2), and whose third and fifth morphisms are the canonical ones constructed in Proposition 2.2, and whose right vertical is defined similarly. Then this diagram commutes. Upon expanding the rows and columns of the diagram, the proof is seen to follow from the trivial commutativity of the diagram

$$\begin{array}{ccc} Fu_{j*}u_{jk*}^{j}u_{ijk*}^{jk} & \xrightarrow{=} & Fu_{i*}u_{ik*}^{i}u_{ijk*}^{ik} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ Fu_{jk*}u_{ijk*}^{jk} & \underset{=}{\longrightarrow} & Fu_{ik*}u_{ijk*}^{ik}. \end{array}$$

Step 5: We show that $\ker\delta \cong \ker\theta_{Fu_{k*}}$. We retain the notation from Step 4. It suffices to show that, for all i, j, l, the diagram

$$\begin{array}{ccc} v_{ik*}^{k}v_{ik}^{i*}\mathcal{F}_i & \xrightarrow{\delta_i^{jl}} & v_{jk*}^{k}v_{jkl*}^{jk}v_{jkl}^{jk*}v_{jk}^{j*}\mathcal{F}_j \\ \downarrow & & \downarrow \\ v_{ik*}^{k}\mathcal{E}_i & \underset{(\theta_{Fu_{k*}})_i^{jl}}{\longrightarrow} & v_{jk*}^{k}v_{jkl*}^{jk}v_{jkl}^{jk*}\mathcal{E}_j \end{array} \tag{32}$$

whose verticals correspond, under the equivalence of Proposition 2.2, to the composition of functors

$$\begin{array}{rcl} - \otimes_{\mathcal{O}_{U_{ik}}} v_{ik}^{i*}\mathcal{F}_i & \xrightarrow{\cong} & u_{ik*}^i(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ & \xrightarrow{\cong} & Fu_{i*}u_{ik*}^i \\ & \xrightarrow{=} & Fu_{ik*} \\ & \xrightarrow{\cong} & - \otimes_{\mathcal{O}_{U_{ik}}} \mathcal{E}_i, \end{array}$$

commutes.

If $i \neq j$ and $i \neq l$, both routes of (32) are 0 by definition of δ and θ. If $i = j$, both the top and bottom of (32) are induced by the unit of $(v_{jkl}^{jk*}, v_{jkl*}^{jk})$ so that (32) commutes in this case as well. It remains to prove that the diagram

$$\begin{array}{ccc} v_{jk*}^k v_{jk}^{j*}\mathcal{F}_j & \xrightarrow{\delta_j^{ij}} & v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ik*} v_{ik}^{i*}\mathcal{F}_i \\ \downarrow & & \downarrow \\ v_{jk*}^k \mathcal{E}_j & \underset{(\theta_{Fu_{k*}})_j^{ij}}{\longrightarrow} & v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ik*}\mathcal{E}_i \end{array} \tag{33}$$

whose verticals equal those of (32), commutes.

By Step 3, (33) may be broken up into the diagram

$$\begin{array}{ccc} v_{jk*}^k v_{jk}^{j*}\mathcal{F}_j & \longrightarrow & v_{jk*}^k v_{ijk*}^{jk} v_{ijk}^{jk*} v_{jk}^{j*}\mathcal{F}_j \\ \downarrow & & \downarrow \\ v_{jk*}^k \mathcal{E}_j & \longrightarrow & v_{jk*}^k v_{ijk*}^{jk} v_{ijk}^{jk*}\mathcal{E}_j \end{array} \tag{34}$$

whose horizontals are induced by units, to the left of the diagram

$$\begin{array}{ccc} v_{jk*}^k v_{ijk*}^{jk} v_{ijk}^{jk*} v_{jk}^{j*}\mathcal{F}_j & \longrightarrow & v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ik*} v_{ik}^{i*}\mathcal{F}_i \\ \downarrow & & \downarrow \\ v_{jk*}^k v_{ijk*}^{jk} v_{ijk}^{jk*}\mathcal{E}_j & \longrightarrow & v_{ik*}^k v_{ijk*}^{ik} v_{ijk}^{ik*}\mathcal{E}_i \end{array} \tag{35}$$

which is v_{ijk*}^k applied to (31). The commutativity of (34) is elementary, while the commutativity of (35) follows from Step 4.

Step 6: Retain the notation from Step 5. We prove that there is an isomorphism $\rho : W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \longrightarrow v_k^* W_{\mathfrak{U}}(F)$ *making the diagram*

$$\begin{array}{ccc} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) & \longrightarrow & \oplus_i v_{ik*}^k \mathcal{E}_i \\ & & \downarrow \\ \rho \downarrow & & \oplus_i v_{ik*}^k v_{ik}^{i*}\mathcal{F}_i \\ & & \downarrow \\ v_k^* W_{\mathfrak{U}}(F) & \longrightarrow & \oplus_i v_k^* v_{i*}\mathcal{F}_i \end{array} \tag{36}$$

whose top vertical is induced by the inverse of the left vertical in (32) and whose bottom vertical is induced by basechange, commute. The proof will follow.

By Step 5 there is an isomorphism $\rho_1 : W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \to \ker\delta$ making the diagram

$$\begin{array}{ccc} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) & \longrightarrow & \oplus_i v_{ik*}^k \mathcal{E}_i \\ {\scriptstyle\rho_1}\downarrow & & \downarrow \\ \ker\delta & \longrightarrow & \oplus_i v_{ik*}^k v_{ik}^{i*}\mathcal{F}_i \end{array}$$

whose right vertical is the upper right vertical in (36), commute. By Step 3, there is an isomorphism $\rho_2 : \ker\delta \longrightarrow v_k^* W_{\mathfrak{U}}(F)$ making the diagram

$$\begin{array}{ccc} \ker\delta & \longrightarrow & \oplus_i v_{ik*}^k v_{ik}^{i*}\mathcal{F}_i \\ {\scriptstyle\rho_2}\downarrow & & \downarrow \\ v_k^* W_{\mathfrak{U}}(F) & \longrightarrow & \oplus_i v_k^* v_{i*}\mathcal{F}_i \end{array}$$

whose right vertical is induced by basechange, commute. We let $\rho = \rho_2\rho_1$. Naturality of ρ in F is a straightforward but tedious exercise, which we omit. □

We now work towards a proof of Proposition 5.4. We begin by introducing some notation and proving a preliminary lemma.

Let S be a scheme with finite open cover $\{W_i\}_{i\in I}$ where $I = \{1,\dots,n\}$ and let $\mathcal{F}$ be an object of $\mathsf{Qcoh}S$. Let

$$\psi_{ij} : w_{ij}^{i*}w_i^*\mathcal{F} \xrightarrow{\cong} (w_i w_{ij}^i)^*\mathcal{F} \xrightarrow{=} (w_j w_{ij}^j)^* \xrightarrow{\cong} w_{ij}^{j*}w_j^*\mathcal{F}$$

denote the canonical isomorphism, let

$$\phi_i^{ij} : w_{i*}w_i^*\mathcal{F} \longrightarrow w_{i*}w_{ij*}^i w_{ij}^{i*}w_i^*\mathcal{F}$$

be induced by the unit of (w_{ij}^{i*}, w_{ij*}^i), and let $\phi_j^{ij} = w_{ij*}\psi_{ji}\circ\phi_j^{ji}$. We define

$$\delta_{\mathcal{F}} : \oplus_i w_{i*}w_i^*\mathcal{F} \longrightarrow \oplus_{i<j} w_{i*}w_{ij*}^i w_{ij}^{i*}w_i^*\mathcal{F} \tag{37}$$

via its components

$$\delta_i^{jk} : w_{i*}w_i^*\mathcal{F} \longrightarrow w_{j*}w_{jk*}^j w_{jk}^{j*}w_j^*\mathcal{F}$$

as follows:

$$\delta_i^{jk} = \begin{cases} \phi_i^{ik} & \text{if } i = j, \\ -\phi_i^{ji} & \text{if } i = k, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

Lemma 5.3. *The map $\mathcal{F} \longrightarrow \oplus_i w_{i*}w_i^*\mathcal{F}$ induced by the units of $\{(w_i^*, w_{i*})\}_i$ is a kernel of $\delta_{\mathcal{F}}$.*

Proof. Let $\eta_{\mathcal{F}} : \mathcal{F} \to \oplus_i w_{i*}w_i^*\mathcal{F}$ be induced by the units of $\{(w_i^*, w_{i*})\}_i$ and let

$$\phi : \mathcal{N} \to \oplus_i w_{i*}w_i^*\mathcal{F}$$

be a morphism such that $\delta_{\mathcal{F}}\phi = 0$. We must show that there exists a unique $\psi : \mathcal{N} \to \mathcal{F}$ such that $\eta_{\mathcal{F}}\psi = \phi$.

For each i, ϕ has a component

$$\phi_i : \mathcal{N} \to w_{i*}w_i^*\mathcal{F}.$$

By adjointness of (w_i^*, w_{i*}), there exists a morphism

$$\psi_i : w_i^*\mathcal{N} \to w_i^*\mathcal{F}$$

such that ϕ_i is the composition

$$\mathcal{N} \longrightarrow w_{i*}w_i^*\mathcal{N} \xrightarrow{w_{i*}\psi_i} w_{i*}w_i^*\mathcal{F}.$$

whose left map is the unit.

Step 1: We show that there exists a unique morphism $\psi : \mathcal{N} \to \mathcal{F}$ such that $w_i^*\psi = \psi_i$ for all i. It suffices, by [1, Section 6.1], to show that, for all pairs i, j, the diagram

$$\begin{array}{ccccccc} (w_iw_{ij}^i)^*\mathcal{N} & \xrightarrow{\cong} & w_{ij}^{i*}w_i^*\mathcal{N} & \xrightarrow{w_{ij}^{i*}\psi_i} & w_{ij}^{i*}w_i^*\mathcal{F} & \xrightarrow{\cong} & (w_iw_{ij}^i)^*\mathcal{F} \\ = \downarrow & & & & & & \downarrow = \\ (w_jw_{ij}^j)^*\mathcal{N} & \xrightarrow[\cong]{} & w_{ij}^{j*}w_j^*\mathcal{N} & \underset{w_{ij}^{j*}\psi_j}{\longrightarrow} & w_{ij}^{j*}w_j^*\mathcal{F} & \xrightarrow[\cong]{} & (w_jw_{ij}^j)^*\mathcal{F} \end{array} \tag{39}$$

whose unlabelled arrows are canonical, commutes. To this end, we note that since $\delta_F\phi = 0$, the diagram

$$\begin{array}{ccccccc} \mathcal{N} & \longrightarrow & w_{i*}w_i^*\mathcal{N} & \xrightarrow{w_{i*}\psi_i} & w_{i*}w_i^*\mathcal{F} & \longrightarrow & w_{i*}w_{ij*}^iw_{ij}^{i*}w_{i*}\mathcal{F} \\ = \downarrow & & & & & & \downarrow \cong \\ \mathcal{N} & \longrightarrow & w_{j*}w_j^*\mathcal{N} & \underset{w_{j*}\psi_j}{\longrightarrow} & w_{j*}w_j^*\mathcal{F} & \longrightarrow & w_{j*}w_{ij*}^jw_{ij}^{j*}w_j^*\mathcal{F} \end{array} \tag{40}$$

whose right vertical is canonical and whose other unlabelled morphisms are units, commutes for all pairs i, j.

Applying w_{ij}^* to (40) yields the commutative diagram

$$\begin{array}{ccccccc} w_{ij}^*\mathcal{N} & \longrightarrow & w_{ij}^*w_{i*}w_i^*\mathcal{N} & \xrightarrow{w_{ij}^*w_{i*}\psi_i} & w_{ij}^*w_{i*}w_i^*\mathcal{F} & \longrightarrow & w_{ij}^*w_{i*}w_{ij*}^iw_{ij}^{i*}w_{i*}\mathcal{F} \\ = \downarrow & & & & & & \downarrow \cong \\ w_{ij}^*\mathcal{N} & \longrightarrow & w_{ij}^*w_{j*}w_j^*\mathcal{N} & \underset{w_{ij}^*w_{j*}\psi_j}{\longrightarrow} & w_{ij}^*w_{j*}w_j^*\mathcal{F} & \longrightarrow & w_{ij}^*w_{j*}w_{ij*}^jw_{ij}^{j*}w_j^*\mathcal{F}. \end{array} \tag{41}$$

Consider the following diagram

$$\begin{array}{ccccc} w_{ij}^{i*}w_i^*w_{i*}w_i^*\mathcal{N} & \xrightarrow{w_{ij}^{i*}w_i^*w_{i*}\psi_i} & w_{ij}^{i*}w_i^*w_{i*}w_i^*\mathcal{F} & \longrightarrow & w_{ij}^{i*}w_i^*\mathcal{F} \\ \uparrow & & \downarrow & & \downarrow = \\ w_{ij}^{i*}w_i^*\mathcal{N} & & w_{ij}^{i*}w_i^*w_{i*}w_{ij*}^iw_{ij}^{i*}w_i^*\mathcal{F} & \longrightarrow & w_{ij}^{i*}w_i^*\mathcal{F} \\ \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ w_{ij}^{j*}w_j^*\mathcal{N} & & w_{ij}^{j*}w_j^*w_{j*}w_{ij*}^jw_{ij}^{j*}w_j^*\mathcal{F} & \longrightarrow & w_{ij}^{j*}w_j^*\mathcal{F} \\ \downarrow & & \uparrow & & \downarrow = \\ w_{ij}^{j*}w_j^*w_{j*}w_j^*\mathcal{N} & \xrightarrow[w_{ij}^{j*}w_j^*w_{j*}\psi_j]{} & w_{ij}^{j*}w_j^*w_{j*}w_j^*\mathcal{F} & \longrightarrow & w_{ij}^{j*}w_j^*\mathcal{F} \end{array} \tag{42}$$

whose unadorned arrows are induced by units and counits, and whose unlabelled isomorphisms are canonical. It follows from a straightforward computation that the commutativity of (41) implies the commutativity of (42). As one can check, the outside circuit of this diagram starting at $w_{ij}^{i*}w_i^*\mathcal{N}$ equals (39).

Step 2: We show that the map $\psi : \mathcal{N} \to \mathcal{F}$ from Step 1 is unique such that the diagram

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & w_{i*}w_i^*\mathcal{N} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle w_{i*}\psi_i} \\ \mathcal{F} & \longrightarrow & w_{i*}w_i^*\mathcal{F} \end{array} \tag{43}$$

whose horizontals are units, commutes for all i. We first note that ψ makes (43) commute by naturality of the unit of (w_i^*, w_{i*}), since $\psi = w_i^*\psi_i$.

We next note that if $\gamma : \mathcal{N} \to \mathcal{F}$ replacing ψ in (43) makes (43) commute for all i, the commutativity of the diagram constructed by applying w_i^* to (43) and composing on the right with the counit $w_i^*w_{i*} \to \mathrm{id}_{\mathsf{Qcoh}S}$ implies that $w_i^*\gamma = \psi_i$. Step 1 tells us that ψ is unique with this property. Therefore $\gamma = \psi$.

Step 3: We complete the proof. By Step 2, $\psi : \mathcal{N} \to \mathcal{F}$ is unique making the diagram

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \oplus_i w_{i*}w_i^*\mathcal{N} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle w_{i*}\psi_i} \\ \mathcal{F} & \underset{\eta_{\mathcal{F}}}{\longrightarrow} & \oplus_i w_{i*}w_i^*\mathcal{F}. \end{array}$$

whose top horizontal is induced by units, commute. By the construction of ψ_i, the top route of this diagram is ϕ. The result follows. □

Proposition 5.4. *If $\mathcal{F}$ is an object of the category $\mathsf{Qcoh}X \times Y$ and F is an object of the category $\mathsf{Bimod}_k(X - Y)$ such that $F \cong - \otimes_{\mathcal{O}_X} \mathcal{F}$, then $W_{\mathfrak{U}}(F) \cong \mathcal{F}$.*

Proof. Since $W_{\mathfrak{U}}$ is a functor, we may assume without loss of generality that $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$. Let $\psi_i : \mathcal{F}_i \longrightarrow v_i^*\mathcal{F}$ correspond, via Proposition 2.2, to the composition

$$\begin{aligned} - \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i &\xrightarrow{\cong} Fu_{i*} \\ &\xrightarrow{=} u_{i*}(-) \otimes_{\mathcal{O}_X} \mathcal{F} \\ &\xrightarrow{\cong} - \otimes_{\mathcal{O}_{U_i}} v_i^*\mathcal{F} \end{aligned}$$

whose first arrow is the canonical isomorphism from the proof of Proposition 2.2, and whose third arrow is (2).

By Lemma 5.3, it suffices to prove that the diagram

$$\begin{array}{ccc} \oplus_i v_{i*}v_i^*\mathcal{F} & \xrightarrow{\delta_F} & \oplus_{i<j} v_{i*}v_{ij*}^i v_{ij}^{i*} v_i^*\mathcal{F} \\ {\scriptstyle\oplus_i v_{i*}\psi_i^{-1}}\downarrow & & \downarrow{\scriptstyle\oplus_{i<j} v_{i*}v_{ij*}^i v_{ij}^{i*}\psi_i^{-1}} \\ \oplus_i v_{i*}\mathcal{F}_i & \underset{\theta_F}{\longrightarrow} & \oplus_{i<j} v_{i*}v_{ij*}^i v_{ij}^{i*}\mathcal{F}_i \end{array}$$

commutes, where we specialize the notation for the definition of δ_F preceding Lemma 5.3 to our situation by setting $S = X \times Y$, $W_i = U_i \times Y$, and $w_i = v_i$.

We recall that δ_i^{ij} denotes the component of $\delta_{\mathcal{F}}$ from the ith summand to the i,jth summand, and θ_i^{ij} is defined similarly. The verification that

$$v_{i*}v_{ij*}^i v_{ij}^{i*}\psi_i^{-1} \circ \delta_i^{ij} = \theta_i^{ij} \circ v_{i*}\psi_i^{-1}$$

is trivial, so that it remains to check that the diagram

$$\begin{array}{ccccc} v_{j*}\mathcal{F}_j & \longrightarrow & v_{j*}v_{ij*}^{j}v_{ij}^{j*}\mathcal{F}_j & \xrightarrow{v_{ij*}\psi_{ji}} & v_{i*}v_{ij*}^{i}v_{ij}^{i*}\mathcal{F}_i \\ {\scriptstyle v_{j*}\psi_j}\downarrow & & {\scriptstyle v_{j*}v_{ij*}^{j}v_{ij}^{j*}\psi_j}\downarrow & & \downarrow{\scriptstyle v_{i*}v_{ij*}^{i}v_{ij}^{i*}\psi_i} \\ v_{j*}v_j^*\mathcal{F} & \longrightarrow & v_{j*}v_{ij*}^{j}v_{ij}^{j*}v_j^*\mathcal{F} & \underset{\cong}{\longrightarrow} & v_{i*}v_{ij*}^{i}v_{ij}^{i*}v_i^*\mathcal{F} \end{array}$$

whose unadorned arrows are induced by units, whose unlabelled isomorphism is canonical, and whose upper right horizontal is defined by (22), commutes. The left square commutes by naturality of units, while to prove the right square commutes, it suffices to prove that the square

$$\begin{array}{ccc} v_{ij}^{j*}\mathcal{F}_j & \xrightarrow{\psi_{ji}} & v_{ij}^{i*}\mathcal{F}_i \\ {\scriptstyle v_{ij}^{j*}\psi_j}\downarrow & & \downarrow{\scriptstyle v_{ij}^{i*}\psi_i} \\ v_{ij}^{j*}v_j^*\mathcal{F} & \underset{\cong}{\longrightarrow} & v_{ij}^{i*}v_i^*\mathcal{F} \end{array} \tag{44}$$

whose unlabeled isomorphism is canonical, commutes. To prove that (44) commutes, it suffices, by Proposition 2.2, to prove that the diagram resulting in applying the functor $-\otimes_{\mathcal{O}_{U_{ij}}}(-)$ to (44) commutes. Upon expanding the resulting diagram, it is straightforward to check that the commutativity of (44) follows from the commutativity of the diagram

$$\begin{array}{ccccc} -\otimes_{\mathcal{O}_{U_{ij}}} v_{ij}^{j*}v_j^*\mathcal{F} & \xrightarrow{\cong} & u_{ij*}^{j}(-)\otimes_{\mathcal{O}_{U_j}} v_j^*\mathcal{F} & \xrightarrow{\cong} & u_{j*}u_{ij*}^{j}(-)\otimes_{\mathcal{O}_X}\mathcal{F} \\ \cong\downarrow & & & & \downarrow= \\ -\otimes_{\mathcal{O}_{U_{ij}}} v_{ij}^{i*}v_i^*\mathcal{F} & \underset{\cong}{\longrightarrow} & u_{ij*}^{i}(-)\otimes_{\mathcal{O}_{U_i}} v_i^*\mathcal{F} & \underset{\cong}{\longrightarrow} & u_{i*}u_{ij*}^{i}(-)\otimes_{\mathcal{O}_X}\mathcal{F} \end{array} \tag{45}$$

whose left vertical is canonical and whose horizontal isomorphisms are induced by the inverse of (2). The commutativity of (45) follows from Lemma 3.2. □

Corollary 5.5. *If $F\in \mathsf{Bimod}_k(X-Y)$ is exact, then $\mathrm{pr}_1^*(-)\otimes_{\mathcal{O}_{X\times Y}} W_{\mathfrak{U}}(F)$ is exact.*

Proof. We first claim that $-\otimes_{\mathcal{O}_{U_i}} v_i^*W_{\mathfrak{U}}(F)$ is exact. To prove the claim, we note that by Proposition 2.2, $Fu_{i*}\cong -\otimes_{\mathcal{O}_{U_i}}\mathcal{F}$ for some quasi-coherent $\mathcal{O}_{U_i\times Y}$-module $\mathcal{F}$. Thus, by Proposition 5.4 and Proposition 5.2, $Fu_{i*}\cong -\otimes_{\mathcal{O}_{U_i}} v_i^*W_{\mathfrak{U}}(F)$. The claim follows.

We now proceed to prove the corollary. Let $p,q: U_i\times Y\to U_i, Y$ denote projections. It suffices to show that, for all i, $v_i^*(\mathrm{pr}_1^*(-)\otimes_{\mathcal{O}_{X\times Y}} W_{\mathfrak{U}}(F))$ is exact. We note that

$$\begin{aligned} v_i^*(\mathrm{pr}_1^*(-)\otimes_{\mathcal{O}_{X\times Y}} W_{\mathfrak{U}}(F)) &\cong v_i^*\,\mathrm{pr}_1^*(-)\otimes_{\mathcal{O}_{U_i\times Y}} v_i^*W_{\mathfrak{U}}(F) \\ &\cong p^*u_i^*(-)\otimes_{\mathcal{O}_{U_i\times Y}} v_i^*W_{\mathfrak{U}}(F). \end{aligned}$$

Thus, to complete the proof, it suffices to show that if $\phi:\mathcal{M}\to\mathcal{N}$ is monic and $V\subset Y$ is affine open, then $q_*(p^*u_i^*(\phi)\otimes_{\mathcal{O}_{U_i\times Y}} v_i^*W_{\mathfrak{U}}(F))(V)$ is monic. But $q_*(p^*u_i^*(-)\otimes_{\mathcal{O}_{U_i\times Y}} v_i^*W_{\mathfrak{U}}(F))$ is exact by the claim and the corollary follows. □

6. The Eilenberg-Watts Transformation

Our goal in this section is to prove the generalization of the Eilenberg-Watts Theorem mentioned in Section 1 (Theorem 1.4). Throughout this section, we use the fact that since X is separated, every object of $\mathsf{Qcoh}X$ is a quotient of a flat object [8, Lemma 1.1.4]. We begin by constructing, for each F in $\mathsf{Bimod}_k(X-Y)$, a natural transformation

$$\Gamma_F : F \longrightarrow - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

which we show is natural in F.

The construction of Γ_F will allow us to describe obstructions to its being an isomorphism (Corollary 6.2). It will also follow readily from the construction of Γ_F that if $F \cong - \otimes_{\mathcal{O}_X} \mathcal{F}$ for some object $\mathcal{F}$ in $\mathsf{Qcoh}X \times Y$ then Γ_F is an isomorphism (Proposition 6.4), and Γ is compatible with affine localization (Proposition 6.6). As a consequence of this last property, we show that the kernel and cokernel of Γ_F are totally global (Corollary 6.7). It follows immediately that Γ_F is an isomorphism if X is affine or if F is exact.

6.1. **Construction of the Eilenberg-Watts Transformation.** Let F be an object of $\mathsf{Bimod}_k(X-Y)$. We construct a natural transformation

$$\Gamma_F : F \longrightarrow - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

and show it is natural in F.

Step 1: We note that for any morphism $\lambda : \mathcal{M} \longrightarrow \mathcal{N}$ in $\mathsf{Qcoh}X$, the canonical morphism coming from the universal property of the kernel

$$\pi : F(\ker \lambda) \longrightarrow \ker F\lambda$$

is natural in the sense that if

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\lambda} & \mathcal{N} \\ \downarrow & & \downarrow \\ \mathcal{M}' & \xrightarrow[\lambda']{} & \mathcal{N}' \end{array}$$

commutes, then the induced maps $\iota : F(\ker \lambda) \longrightarrow F(\ker \lambda')$ and $\iota' : \ker F\lambda \longrightarrow \ker F\lambda'$ make the diagram

$$\begin{array}{ccc} F(\ker \lambda) & \longrightarrow & \ker F\lambda \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \iota'} \\ F(\ker \lambda') & \longrightarrow & \ker F\lambda' \end{array}$$

whose horizontals are the canonical morphisms, commute.

Step 2: Let $\mathcal{L}$ be a flat object in $\mathsf{Qcoh}X$. We construct a morphism

$$\Gamma_{F\mathcal{L}} : F(\mathcal{L}) \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

in the category $\mathsf{Qcoh}Y$. Specialize the notation preceding Lemma 5.3 to the case that $S = X$ and $W_i = U_i$. By Lemma 5.3, the morphism

$$\mathcal{L} \longrightarrow \oplus_i u_{i*}u_i^*\mathcal{L}$$

induced by unit morphisms is a kernel of

$$\delta_{\mathcal{L}} : \oplus_i u_{i*}u_i^*\mathcal{L} \longrightarrow \oplus_{i<j} u_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L}.$$

Let

$$\pi_1 : F(\mathcal{L}) \longrightarrow \ker F(\delta_{\mathcal{L}})$$

denote the morphism from Step 1. Let γ_i denote the composition

$$Fu_{i*}u_i^*\mathcal{L} \longrightarrow u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i$$

whose left arrow is the canonical isomorphism from Proposition 2.2 and whose right arrow is induced by (1). Let γ_{ij} denote the composition

$$\begin{aligned} Fu_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} &\xrightarrow{\cong} u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ &\xrightarrow{\cong} u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_{ij}}} v^{i*}_{ij}\mathcal{F}_i \\ &\xrightarrow{\cong} u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\ &\xrightarrow{\cong} \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \end{aligned}$$

whose first arrow is from Proposition 2.2, whose second arrow is induced by (2) and whose third and fourth arrows are induced by (1).

We first claim

$$\gamma_{ij} \circ F\delta_i^{ij} = \mathcal{L} \otimes_{\mathcal{O}_X} \theta_i^{ij} \circ \gamma_i, \tag{46}$$

for all $i < j$, where θ_i^{ij} is defined in (23). To prove the claim, consider the following diagram

$$\begin{array}{ccc} Fu_{i*}u_i^*\mathcal{L} & \longrightarrow & Fu_{i*}u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} \\ \downarrow & & \downarrow \\ u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i & \longrightarrow & u^i_{ij*}u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i & & u^{i*}_{ij}u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_{ij}}} v^{i*}_{ij}\mathcal{F}_i \\ \downarrow & & \downarrow \\ \mathcal{L} \otimes_{\mathcal{O}_X} v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i & \longrightarrow & u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \end{array}$$

whose two top horizontals and bottom-left vertical are induced by the units, whose top verticals are from Proposition 2.2, whose left-middle vertical is induced by (1), whose right-middle vertical is induced by (2), whose right-bottom vertical is induced by (1) and whose bottom horizontal is induced by the inverse of (1). The claim will follow from the commutativity of this diagram. The top square commutes by the naturality of the canonical isomorphism from Proposition 2.2. To show that the bottom rectangle commutes, we first split it down the diagonal via the morphism

$$u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \longrightarrow u_i^*\mathcal{L} \otimes_{\mathcal{O}_{U_i}} v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i$$

induced by the unit of (v^{i*}_{ij}, v^i_{ij*}). The resulting left subdiagram commutes by the naturality of (1), while the right subdiagram commutes by the commutativity of (3).

We next claim

$$\gamma_{ij} \circ F\delta_j^{ij} = \mathcal{L} \otimes_{\mathcal{O}_X} \theta_j^{ij} \circ \gamma_j.$$

To prove the claim, consider the following diagram

$$\begin{array}{ccc}
Fu_{j*}u^j_{ij*}u^{j*}_{ij}u^*_j\mathcal{L} & \longrightarrow & Fu_{i*}u^i_{ij*}u^{i*}_{ij}u^*_i\mathcal{L} \\
\downarrow & & \downarrow \\
u^j_{ij*}u^{j*}_{ij}u^*_j\mathcal{L}\otimes_{\mathcal{O}_{U_j}}\mathcal{F}_j & & u^i_{ij*}u^{i*}_{ij}u^*_i\mathcal{L}\otimes_{\mathcal{O}_{U_i}}\mathcal{F}_i \\
\downarrow & & \downarrow \\
u^{j*}_{ij}u^*_j\mathcal{L}\otimes_{\mathcal{O}_{U_{ij}}}v^{j*}_{ij}\mathcal{F}_j & & u^{i*}_{ij}u^*_i\mathcal{L}\otimes_{\mathcal{O}_{U_{ij}}}v^{i*}_{ij}\mathcal{F}_i \\
\downarrow & & \downarrow \\
u^*_j\mathcal{L}\otimes_{\mathcal{O}_{U_j}}v^j_{ij*}v^{j*}_{ij}\mathcal{F}_j & & u^*_i\mathcal{L}\otimes_{\mathcal{O}_{U_i}}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\
\downarrow & & \downarrow \\
\mathcal{L}\otimes_{\mathcal{O}_X}v_{j*}v^j_{ij*}v^{j*}_{ij}\mathcal{F}_j & \longrightarrow & \mathcal{L}\otimes_{\mathcal{O}_X}v_{j*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i
\end{array}$$

whose top horizontal is induced by the canonical isomorphism

$$u^{j*}_{ij}u^*_j\xrightarrow{\cong}(u_ju^j_{ij})^*\xrightarrow{=}(u_iu^i_{ij})^*\xrightarrow{\cong}u^{i*}_{ij}u^*_i$$

whose verticals are induced by (2) and (1), and whose bottom horizontal is induced by the map

$$\psi_{ji}:v^{j*}_{ij}\mathcal{F}_j\xrightarrow{\cong}v^{i*}_{ij}\mathcal{F}_i$$

defined after (22). Since (46) holds when i and j are interchanged, the proof of the claim follows from the commutativity of this diagram. This follows easily from the definition of ψ_{ji}.

Next, consider the following diagram

$$\begin{array}{ccc}
F(\oplus_iu_{i*}u^*_i\mathcal{L}) & \xrightarrow{F\delta_{\mathcal{L}}} & F(\oplus_{i<j}u_{i*}u^i_{ij*}u^{i*}_{ij}u^*_i\mathcal{L}) \\
\downarrow & & \downarrow \\
\oplus_iFu_{i*}u^*_i\mathcal{L} & \longrightarrow & \oplus_{i<j}Fu_{i*}u^i_{ij*}u^{i*}_{ij}u^*_i\mathcal{L} \\
{\scriptstyle\oplus_i\gamma_i}\downarrow & & \downarrow{\scriptstyle\oplus_{i<j}\gamma_{ij}} \\
\oplus_i\mathcal{L}\otimes_{\mathcal{O}_X}v_{i*}\mathcal{F}_i & \longrightarrow & \oplus_{i<j}\mathcal{L}\otimes_{\mathcal{O}_X}v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i \\
\downarrow & & \downarrow \\
\mathcal{L}\otimes_{\mathcal{O}_X}(\oplus_iv_{i*}\mathcal{F}_i) & \underset{\mathcal{L}\otimes_{\mathcal{O}_X}\theta_F}{\longrightarrow} & \mathcal{L}\otimes_{\mathcal{O}_X}(\oplus_{i<j}v_{i*}v^i_{ij*}v^{i*}_{ij}\mathcal{F}_i)
\end{array}\tag{47}$$

whose second horizontal is induced by the maps $F\delta^{jk}_i$, whose third horizontal is induced by the maps $\mathcal{L}\otimes_{\mathcal{O}_X}\theta^{jk}_i$ and whose corner verticals are canonical isomorphisms. It follows from the claims that the center square in the diagram commutes. Since the top and bottom square of (47) commute, there is an induced isomorphism

$$\pi_2:\ker F\delta_{\mathcal{L}}\xrightarrow{\cong}\ker(\mathcal{L}\otimes_{\mathcal{O}_X}\theta_F).$$

Finally, since $\mathcal{L}$ is flat and pr_{2*} is left-exact, there is a canonical isomorphism

$$\pi_3:\ker(\mathcal{L}\otimes_{\mathcal{O}_X}\theta_F)\xrightarrow{\cong}\mathcal{L}\otimes_{\mathcal{O}_X}\ker\theta_F.$$

We define

$$\Gamma_{F\mathcal{L}} = \pi_3\pi_2\pi_1.$$

Step 3: We show Γ_F is natural on flats, i.e. we show that if

$$\psi : \mathcal{L} \longrightarrow \mathcal{L}'$$

is a morphism of flat objects in $\mathsf{Qcoh}X$ *then the diagram*

$$\begin{array}{ccc} F\mathcal{L} & \xrightarrow{F\psi} & F\mathcal{L}' \\ {\scriptstyle \Gamma_{F\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle \Gamma_{F\mathcal{L}'}} \\ \mathcal{L}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \xrightarrow[\psi\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)]{} & \mathcal{L}'\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{array} \tag{48}$$

commutes. We leave it as an easy exercise for the reader to check that the diagram

$$\begin{array}{ccc} F(\oplus_i u_{i*}u_i^*\mathcal{L}) & \xrightarrow{F\delta_{\mathcal{L}}} & F(\oplus_{i<j} u_{i*}u_{ij*}^i u_{ij}^{i*}u_i^*\mathcal{L}) \\ \downarrow & & \downarrow \\ F(\oplus_i u_{i*}u_i^*\mathcal{L}') & \xrightarrow[F\delta_{\mathcal{L}'}]{} & F(\oplus_{i<j} u_{i*}u_{ij*}^i u_{ij}^{i*}u_i^*\mathcal{L}') \end{array}$$

whose verticals are induced by ψ, commutes. Therefore, by Step 1, the induced morphism $\psi' : \ker F\delta_{\mathcal{L}} \longrightarrow \ker F\delta_{\mathcal{L}'}$ makes the diagram

$$\begin{array}{ccc} F\mathcal{L} = F\ker\delta_{\mathcal{L}} & \longrightarrow & \ker F\delta_{\mathcal{L}} \\ {\scriptstyle F\psi}\downarrow & & \downarrow{\scriptstyle \psi'} \\ F\mathcal{L}' = F\ker\delta_{\mathcal{L}'} & \longrightarrow & \ker F\delta_{\mathcal{L}'} \end{array}$$

whose horizontals are canonical, commute. Thus, the top square in the diagram

$$\begin{array}{ccc} F\ker\delta_{\mathcal{L}} & \xrightarrow{F\psi} & F\ker\delta_{\mathcal{L}'} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \ker F\delta_{\mathcal{L}} & \xrightarrow[\psi']{} & \ker F\delta_{\mathcal{L}'} \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \ker(\mathcal{L}\otimes_{\mathcal{O}_X}\theta_F) & \longrightarrow & \ker(\mathcal{L}'\otimes_{\mathcal{O}_X}\theta_F) \\ {\scriptstyle \pi_3}\downarrow & & \downarrow{\scriptstyle \pi_3} \\ \mathcal{L}\otimes_{\mathcal{O}_X}\ker\theta_F & \longrightarrow & \mathcal{L}'\otimes_{\mathcal{O}_X}\ker\theta_F \end{array}$$

whose verticals are defined in Step 2 and whose bottom two horizontals are induced by ψ, commutes. The proofs that the middle and bottom squares of this diagram commute are left as straightforward exercises.

Step 4: We show that, for each $\mathcal{M}$ in $\mathsf{Qcoh}X$ *and each flat presentation*

$$\mathcal{L}_1 \xrightarrow{\xi_1} \mathcal{L}_0 \xrightarrow{\xi_0} \mathcal{M}, \tag{49}$$

there exists a unique morphism

$$\gamma_{F\mathcal{M}} : F\mathcal{M} \longrightarrow \mathcal{M}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

making

$$\begin{array}{ccc} F\mathcal{L}_0 & \xrightarrow{F\xi_0} & F\mathcal{M} \\ {\scriptstyle \Gamma_{F\mathcal{L}_0}}\downarrow & & \downarrow{\scriptstyle \gamma_{F\mathcal{M}}} \\ \mathcal{L}_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \underset{\xi_0\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)}{\longrightarrow} & \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{array} \tag{50}$$

commute. Applying F to the flat presentation (49) yields the first row in the diagram

$$\begin{array}{ccccc} F\mathcal{L}_1 & \xrightarrow{F\xi_1} & F\mathcal{L}_0 & \xrightarrow{F\xi_0} & F\mathcal{M} \\ {\scriptstyle \Gamma_{F\mathcal{L}_1}}\downarrow & & \downarrow{\scriptstyle \Gamma_{F\mathcal{L}_0}} & & \\ \mathcal{L}_1 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \underset{\xi_1\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)}{\longrightarrow} & \mathcal{L}_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \underset{\xi_0\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)}{\longrightarrow} & \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{array} \tag{51}$$

which commutes by Step 3. Thus, there exists a unique morphism

$$\gamma_{F\mathcal{M}} : F\mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$$

making (50) commute.

We will show, in Step 6, that $\gamma_{F\mathcal{M}}$ is independent of presentation chosen.

Step 5: We show that if $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism in $\mathsf{Qcoh}X$, then the diagram

$$\begin{array}{ccc} F\mathcal{M} & \xrightarrow{F\phi} & F\mathcal{N} \\ {\scriptstyle \gamma_{F\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \gamma_{F\mathcal{N}}} \\ \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}} & \underset{\phi\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)}{\longrightarrow} & \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}} \end{array}$$

commutes. Suppose

$$\mathcal{L}_1' \longrightarrow \mathcal{L}_0' \xrightarrow{\pi'} \mathcal{N}$$

is a flat presentation for $\mathcal{N}$ and let $\gamma_{F\mathcal{N}} : F\mathcal{N} \longrightarrow \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ denote the corresponding morphism constructed in Step 4. Then there exists a flat presentation

$$\mathcal{L} \longrightarrow \mathcal{L}_0 \oplus \mathcal{L}_0' \xrightarrow{\phi\pi\oplus\pi'} \mathcal{N} \tag{52}$$

for $\mathcal{N}$, and the corresponding morphism $\gamma'_{F\mathcal{N}}$ constructed in Step 4 makes the outer circuit of the diagram

(53)

$$\begin{array}{ccccc} F(\mathcal{L}_0 \oplus \mathcal{L}_0') & \xrightarrow{F(\pi\oplus \mathrm{id}_{\mathcal{L}_0'})} & F(\mathcal{M} \oplus \mathcal{L}_0') & \xrightarrow{F(\phi\oplus\pi')} & F\mathcal{N} \\ {\scriptstyle \Gamma_{F(\mathcal{L}_0\oplus\mathcal{L}_0')}}\downarrow & & {\scriptstyle \Gamma_{F(\mathcal{M}\oplus\mathcal{L}_0')}}\downarrow & & \downarrow{\scriptstyle \gamma'_{F\mathcal{N}}} \\ (\mathcal{L}_0 \oplus \mathcal{L}_0') \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \longrightarrow & (\mathcal{M} \oplus \mathcal{L}_0') \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \longrightarrow & \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{array}$$

whose bottom-left horizontal is induced by $\pi \oplus \mathrm{id}_{\mathcal{L}_0'}$ and whose bottom-right horizontal is induced by $\phi \oplus \pi'$, commute.

It follows from the commutativity of the outer circuit of (53) and from Step 3 that the outer circuit of the diagram constructed by placing the diagram

(54)
$$\begin{array}{ccc} F\mathcal{L}_0 \oplus F\mathcal{L}'_0 & \stackrel{F\pi\oplus \mathrm{id}_{F\mathcal{L}'_0}}{\longrightarrow} & F\mathcal{M} \oplus F\mathcal{L}'_0 \\ {\scriptstyle \Gamma_{F\mathcal{L}_0}\oplus\Gamma_{F\mathcal{L}'_0}}\downarrow & & {\scriptstyle \gamma_{F\mathcal{M}}\oplus\Gamma_{F\mathcal{L}'_0}}\downarrow \\ (\mathcal{L}_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)) \oplus (\mathcal{L}'_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)) & \longrightarrow & (\mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)) \oplus (\mathcal{L}'_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)) \end{array}$$

whose bottom horizontal is $(\pi \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)) \oplus (\mathrm{id}_{\mathcal{L}'_0} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F))$, to the left of the diagram

(55)
$$\begin{array}{ccc} F\mathcal{M} \oplus F\mathcal{L}'_0 & \stackrel{F\phi\oplus F\pi'}{\longrightarrow} & F\mathcal{N} \\ {\scriptstyle \gamma_{F\mathcal{M}}\oplus\Gamma_{F\mathcal{L}'_0}}\downarrow & & \downarrow{\scriptstyle \gamma'_{F\mathcal{N}}} \\ (\mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)) \oplus (\mathcal{L}'_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)) & \longrightarrow & \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{array}$$

whose bottom horizontal is induced by $\phi \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ and $\pi' \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$, commutes. We note also that the diagram (54) commutes since Step 4 implies that (50) commutes. Since the top horizontal in (54) is an epimorphism, it follows that (55) commutes as well. By restricting both routes of (55) to $F\mathcal{L}'_0$ and using the fact, established in Step 4, that $\gamma_{F\mathcal{N}}$ is unique making the diagram

$$\begin{array}{ccc} F\mathcal{L}'_0 & \stackrel{F\pi'}{\longrightarrow} & F\mathcal{N} \\ {\scriptstyle \Gamma_{F\mathcal{L}'_0}}\downarrow & & \downarrow{\scriptstyle \gamma_{F\mathcal{N}}} \\ \mathcal{L}'_0 \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \underset{\pi'\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)}{\longrightarrow} & \mathcal{N} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{array}$$

commute, we have $\gamma_{F\mathcal{N}} = \gamma'_{F\mathcal{N}}$. On the other hand, restricting both routes of (55) to $F\mathcal{M}$ allows us to conclude that

$$(\phi \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F))\gamma_{F\mathcal{M}} = \gamma'_{F\mathcal{N}} F\phi.$$

Step 5 follows.

Step 6: We show that $\gamma_{F\mathcal{M}}$ is independent of presentation. Let $\gamma'_{F\mathcal{M}} : F\mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ denote the morphism constructed in Step 4 using a flat presentation

$$\mathcal{L}'_1 \longrightarrow \mathcal{L}'_0 \longrightarrow \mathcal{M}.$$

Now apply Step 5 to conclude that the diagram

$$\begin{array}{ccc} F\mathcal{M} & \stackrel{F\,\mathrm{id}_{\mathcal{M}}}{\longrightarrow} & F\mathcal{M} \\ {\scriptstyle \gamma_{F\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \gamma'_{F\mathcal{M}}} \\ \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \underset{\mathrm{id}_{\mathcal{M}} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)}{\longrightarrow} & \mathcal{M} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \end{array}$$

commutes. Step 6 follows.

We define

$$\Gamma_{F\mathcal{M}} := \gamma_{F\mathcal{M}}.$$

Step 7: We show that Γ_F is natural in $\mathcal{M}$. This follows from Step 5 in light of the definition of $\Gamma_{F\mathcal{M}}$ given in Step 6.

Step 8: We show Γ_F is natural in F. It suffices to check that if $\mathcal{L}$ is a flat object in $\mathsf{Qcoh}X$ and $\eta : F \to G$ is a morphism in $\mathsf{Bimod}_k(X - Y)$ then the diagram

$$\begin{array}{ccc} F(\mathcal{L}) & \xrightarrow{\eta_{\mathcal{L}}} & G(\mathcal{L}) \\ {\scriptstyle \Gamma_{F\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle \Gamma_{G\mathcal{L}}} \\ \mathcal{L} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \xrightarrow[\mathcal{L}\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(\eta)]{} & \mathcal{L} \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(G) \end{array} \tag{56}$$

commutes. Sufficiency follows from the right exactness of F. The proof that (56) commutes is straightforward, and we omit it.

6.2. **Properties of the Eilenberg-Watts Transformation.** As in the previous subsection, we specialize the notation preceding Lemma 5.3 to the case that $S = X$ and $W_i = U_i$. Let $\mathcal{M}$ be an object in $\mathsf{Qcoh}X$. By Lemma 5.3, the morphism

$$\mathcal{M} \longrightarrow \oplus_i u_{i*}u_i^*\mathcal{M}$$

induced by unit morphisms is a kernel of

$$\delta_{\mathcal{M}} : \oplus_i u_{i*}u_i^*\mathcal{M} \longrightarrow \oplus_{i<j} u_{i*}u_{ij*}^i u_{ij}^{i*}u_i^*\mathcal{M}.$$

Throughout this subsection, F is assumed to be an object in $\mathsf{Bimod}_k(X - Y)$.

Proposition 6.1. *If $\mathcal{L}$ is a flat object in $\mathsf{Qcoh}X$, then $\Gamma_{F\mathcal{L}}$ is an isomorphism if and only if the canonical map $F \ker \delta_{\mathcal{L}} \to \ker F\delta_{\mathcal{L}}$ is an isomorphism.*

Proof. The map $\Gamma_{F\mathcal{L}}$ is a composition of the canonical map $F \ker \delta_{\mathcal{L}} \to \ker F\delta_{\mathcal{L}}$ and two isomorphisms, by Step 2 of the construction of Γ. □

The next result follows from Proposition 6.1 and a straightforward diagram chase.

Corollary 6.2. *If $F \in \mathsf{Bimod}_k(X - Y)$ then Γ_F is an isomorphism if and only if*

1. *for all flat objects $\mathcal{L}$ in $\mathsf{Qcoh}X$, the canonical map $F \ker \delta_{\mathcal{L}} \to \ker F\delta_{\mathcal{L}}$ is an isomorphism, and*
2. *$- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right exact.*

Corollary 6.3. *Let F be a totally global, exact functor such that $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right exact. Then $F = 0$.*

Proof. Since F is exact and $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ is right exact, $F \cong - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$ by Corollary 6.2. Thus, since F is totally global, $F = 0$ by Proposition 4.4. □

Proposition 6.4. *If $F \cong - \otimes_{\mathcal{O}_X} \mathcal{F}$ for some object $\mathcal{F}$ in $\mathsf{Qcoh}X \times Y$, then Γ_F is an isomorphism.*

Proof. By the naturality of Γ (noted in Step 8 of the construction of Γ) we may assume without loss of generality that $F = - \otimes_{\mathcal{O}_X} \mathcal{F}$. By Proposition 5.4, $W_{\mathfrak{U}}(F) \cong \mathcal{F}$. Since F is right exact, so is $- \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F)$. Hence, by Corollary 6.2, it suffices to show that if $\mathcal{L}$ is a flat object in $\mathsf{Qcoh}X$, then the canonical map $F(\mathcal{L}) = F(\ker \delta_{\mathcal{L}}) \longrightarrow \ker F\delta_{\mathcal{L}}$ is an isomorphism. To prove this, we note that in Step 2 of the construction of Γ we constructed an isomorphism

$$\pi_2^{-1}\pi_3^{-1} : (\ker \delta_{\mathcal{L}}) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \ker(\delta_{\mathcal{L}} \otimes_{\mathcal{O}_X} \mathcal{F}). \tag{57}$$

Hence, to complete the proof of the proposition, it suffices to prove that (57) is the canonical map induced by the universal property of the kernel. This fact follows from Lemma 3.1, as one can check. □

Corollary 6.5. *Let $\mathcal{F}'$ be an object of $\mathsf{Qcoh}X \times Y$ such that $F' := - \otimes_{\mathcal{O}_X} \mathcal{F}'$ is an object in $\mathsf{Bimod}_k(X - Y)$. If $\Phi : F \to F'$ is a morphism in $\mathsf{Bimod}_k(X - Y)$, then Φ factors through Γ_F.*

Proof. Since Γ_G is natural in G, the diagram

$$\begin{array}{ccc} F & \xrightarrow{\Phi} & F' \\ {\scriptstyle \Gamma_F}\downarrow & & \downarrow{\scriptstyle \Gamma_{F'}} \\ - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \xrightarrow[-\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(\Phi)]{} & - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F') \end{array}$$

commutes. Since $\Gamma_{F'}$ is an isomorphism by Proposition 6.4, the assertion follows. □

Proposition 6.6. *Let $\Gamma_F * u_{k*}$ denote the horizontal composition of the natural transformations Γ_F and $\mathrm{id}_{u_{k*}}$. Then Γ_F is compatible with affine localization, i.e. the diagram*

$$\begin{array}{ccc} Fu_{k*} & \xrightarrow{\Gamma_{F*u_{k*}}} & u_{k*}(-) \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \\ {\scriptstyle \Gamma_{Fu_{k*}}}\downarrow & & \downarrow \\ - \otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) & \longrightarrow & - \otimes_{\mathcal{O}_{U_k}} v_k^* W_{\mathfrak{U}}(F) \end{array} \tag{58}$$

whose bottom horizontal is induced by the isomorphism constructed in Proposition 5.2 and whose right vertical is induced by the isomorphism (2), commutes for all k.

Proof. We prove the result in several steps.
Step 1: We show that it suffices to prove that (58) commutes when applied to flat objects of $\mathsf{Qcoh}U_k$. For, if $\pi : \mathcal{L} \to \mathcal{M}$ is an epimorphism in $\mathsf{Qcoh}U_k$ where $\mathcal{L}$ flat, then, since the arrows in (58) are natural, and since Fu_{k*} is right exact, Step 1 follows from a standard diagram chase.
Step 2: Consider the following diagram

$$\begin{array}{ccc} F & \longrightarrow & \oplus_i Fu_{i*}u_i^* \\ & & \downarrow \\ {\scriptstyle \Gamma_F}\downarrow & & \oplus_i u_i^*(-) \otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i \\ & & \downarrow \\ - \otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \longrightarrow & \oplus_i - \otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i \end{array} \tag{59}$$

whose top horizontal is induced by a unit, whose top vertical is induced by the canonical isomorphism from Proposition 2.2, whose bottom vertical is induced by (1), and whose bottom horizontal comes from the definition of $W_{\mathfrak{U}}(F)$ as a kernel. We note that this diagram commutes. We first note that (59) commutes on flats by the definition of Γ_F. Now, if $\mathcal{M}$ is an object in $\mathsf{Qcoh}X$, there exists an epimorphism from a flat object $\mathcal{L}$ in $\mathsf{Qcoh}X$ to $\mathcal{M}$. This epimorphism induces a map from (59) applied to $\mathcal{L}$ to (59) applied to $\mathcal{M}$. Since all the arrows in (59) are natural and the induced map $F\mathcal{L} \to F\mathcal{M}$ is an epimorphism, the commutativity of (59) applied to $\mathcal{M}$ follows from a routine diagram chase.

Step 3: Consider the following diagram

$$
\begin{array}{ccc}
Fu_{k*} & \longrightarrow & Fu_{k*}u^k_{ik*}u^{k*}_{ik} \\
\downarrow & & \downarrow{=} \\
Fu_{i*}u_i^*u_{k*} & \longrightarrow & Fu_{i*}u^i_{ik*}u^{k*}_{ik} \\
\downarrow & & \downarrow \\
u_i^*u_{k*}(-)\otimes_{\mathcal{O}_{U_i}}\mathcal{F}_i & \longrightarrow & u^i_{ik*}u^{k*}_{ik}(-)\otimes_{\mathcal{O}_{U_i}}\mathcal{F}_i \\
\downarrow & & \downarrow \\
u_{k*}(-)\otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i & & u^{k*}_{ik}(-)\otimes_{\mathcal{O}_{U_{ik}}} v^{i*}_{ik}\mathcal{F}_i \\
\downarrow & & \downarrow \\
-\otimes_{\mathcal{O}_{U_k}} v_k^*v_{i*}\mathcal{F}_i & \longrightarrow & -\otimes_{\mathcal{O}_{U_k}} v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i
\end{array}
\tag{60}
$$

whose top horizontal and top-left vertical are unit morphisms, whose second verticals are from Propostion 2.2, whose second, third and fourth horizontal are induced by basechange, whose third left-vertical and bottom right-vertical are induced by (1), and whose bottom-left vertical and third right-vertical are induced by (2). Then this diagram commutes. The proof of the commutativity of the top square of (60) is routine and left to the reader. The commutativity of the middle square of (60) follows from the fact that the second verticals are induced by the same natural transformations. The fact that the bottom rectangle in (60) commutes follows from Lemma 3.3.

Step 4: We complete the proof of the proposition. Recall that $\mathcal{E}_i \in \mathsf{Qcoh}U_{ik} \times Y$ denotes the object corresponding to the functor $Fu_{ik*} \in \mathsf{Bimod}_k(U_{ik} - Y)$ in the proof of Proposition 2.2. Consider the following commutative diagram

$$
\begin{array}{ccccc}
-\otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) & \longrightarrow & \oplus_i -\otimes_{\mathcal{O}_{U_k}} v^k_{ik*}\mathcal{E}_i & \longrightarrow & \oplus_i u^{k*}_{ik}(-)\otimes_{\mathcal{O}_{U_{ik}}}\mathcal{E}_i \\
{\scriptstyle \Gamma_{Fu_{k*}}}\uparrow & & & & \downarrow \\
Fu_{k*} & \underset{=}{\longrightarrow} & Fu_{k*} & \longrightarrow & \oplus_i Fu_{k*}u^k_{ik*}u^{k*}_{ik} \\
 & & \downarrow & & \downarrow{=} \\
 & & \oplus_i Fu_{i*}u_i^*u_{k*} & & \oplus_i Fu_{i*}u^i_{ik*}u^{k*}_{ik} \\
{\scriptstyle \Gamma_{F*u_{k*}}}\downarrow & & \downarrow & & \downarrow \\
 & & \oplus_i u_i^*u_{k*}(-)\otimes_{\mathcal{O}_{U_i}}\mathcal{F}_i & & \oplus_i u^i_{ik*}u^{k*}_{ik}(-)\otimes_{\mathcal{O}_{U_i}}\mathcal{F}_i \\
 & & \downarrow & & \downarrow \\
u_{k*}(-)\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) & \longrightarrow & \oplus_i u_{k*}(-)\otimes_{\mathcal{O}_X} v_{i*}\mathcal{F}_i & & \oplus_i u^{k*}_{ik}(-)\otimes_{\mathcal{O}_{U_{ik}}} v^{i*}_{ik}\mathcal{F}_i \\
\downarrow & & \downarrow & & \downarrow \\
-\otimes_{\mathcal{O}_{U_k}} v_k^*W_{\mathfrak{U}}(F) & \longrightarrow & \oplus_i -\otimes_{\mathcal{O}_{U_k}} v_k^*v_{i*}\mathcal{F}_i & \longrightarrow & \oplus_i -\otimes_{\mathcal{O}_{U_k}} v^k_{ik*}v^{i*}_{ik}\mathcal{F}_i
\end{array}
\tag{61}
$$

whose upper and middle-left rectangle are (59), whose lower-right rectangle is (60) and whose lower-left square has verticals induced by (2) and horizontals induced

by the inclusion

$$W_{\mathfrak{U}}(F) \longrightarrow \oplus_i v_{i*}\mathcal{F}_i. \tag{62}$$

It follows from Step 2, Step 3, and the naturality of (2) that all squares in this diagram commute.

Next, we consider the following commutative diagram

$$\begin{array}{ccc} -\otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) \xrightarrow{=} -\otimes_{\mathcal{O}_{U_k}} W_{\mathfrak{U}\cap U_k}(Fu_{k*}) & & \\ & & \uparrow{\scriptstyle \Gamma_{Fu_{k*}}} \\ & & Fu_{k*} \\ {\scriptstyle -\otimes_{\mathcal{O}_{U_k}}\rho}\downarrow & & \downarrow{\scriptstyle \Gamma_{F*u_{k*}}} \\ & & u_{k*}(-)\otimes_{\mathcal{O}_X} W_{\mathfrak{U}}(F) \\ & & \downarrow \\ -\otimes_{\mathfrak{O}_{U_k}} v_k^* W_{\mathfrak{U}}(F) & \underset{=}{\longrightarrow} & -\otimes_{\mathfrak{O}_{U_k}} v_k^* W_{\mathfrak{U}}(F) \end{array} \tag{63}$$

whose bottom-right vertical is induced by (2). The outside of the diagram formed by placing this diagram to the left of (61) commutes by Step 6 of Proposition 5.2. Since (63) equals (58), and since the map

$$-\otimes_{\mathcal{O}_{U_k}} v_k^* W_{\mathfrak{U}}(F) \longrightarrow \oplus_i (-)\otimes_{\mathcal{O}_{U_k}} v_k^* v_{i*}\mathcal{F}_i$$

induced by (62) is monic on flat objects, we conclude, by a straightforward diagram chase on the diagram constructed by placing (63) to the left of (61), that (58) commutes on flat objects. The proposition follows from Step 1. □

Corollary 6.7. *If F is an object of $\mathsf{Bimod}_k(X-Y)$ then $\ker\Gamma_F$ and $\operatorname{cok}\Gamma_F$ are totally global. In particular, if X is affine, then Γ_F is an isomorphism.*

Proof. By Proposition 4.5, it suffices to show that $(\ker\Gamma_F)u_{i*}$ and $(\operatorname{cok}\Gamma_F)u_{i*}$ equal 0 for all i. To this end, we compute

$$\begin{aligned} (\ker\Gamma_F)u_{i*} &= \ker(\Gamma_F * u_{i*}) \\ &\cong \ker\Gamma_{Fu_{i*}} \\ &= 0 \end{aligned}$$

where the second line follows from Proposition 6.6, and the third follows from the fact that since $Fu_{i*} \cong -\otimes_{\mathcal{O}_{U_i}} \mathcal{F}_i$ by Proposition 2.2, $\Gamma_{Fu_{i*}}$ is an isomorphism by Proposition 6.4.

A similar proof establishes the fact that $\operatorname{cok}\Gamma_F$ is totally global.

The last statement follows from the fact that if X is affine, every totally global functor from $\mathsf{Qcoh}X$ is 0. □

From now on, we fix a finite affine open cover $\mathfrak{U}$ of X and write W for $W_{\mathfrak{U}}$.

Corollary 6.8. *If F is an exact functor in $\mathsf{Bimod}_k(X-Y)$, then Γ_F is an isomorphism.*

Proof. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module and let

$$\delta_{\mathcal{M}} : \oplus_i u_{i*}u_i^*\mathcal{M} \longrightarrow \oplus_{i<j} u_{i*}u_{ij*}^i u_{ij}^{i*}u_i^*\mathcal{M}$$

denote the morphism defined by (37). By Proposition 6.6, the natural transformation Γ_F applied to each term of $\delta_{\mathcal{M}}$ is an isomorphism. Thus, the canonical morphism $\ker F(\delta_{\mathcal{M}}) \longrightarrow \ker(\delta_{\mathcal{M}} \otimes_{\mathcal{O}_X} W(F))$ is an isomorphism. Since F is exact, the canonical morphism $F(\ker \delta_{\mathcal{M}}) \longrightarrow \ker F(\delta_{\mathcal{M}})$ is an isomorphism. On the other hand, by Corollary 5.5, F exact implies that $- \otimes_{\mathcal{O}_X} W(F)$ is left exact. Therefore, the canonical morphism $(\ker \delta_{\mathcal{M}}) \otimes_{\mathcal{O}_X} W(F) \longrightarrow \ker(\delta_{\mathcal{M}} \otimes_{\mathcal{O}_X} W(F))$ is an isomorphism. The result now follows from Lemma 5.3. □

7. A Structure Theorem for Totally Global Functors in $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$

The purpose of this section is to compute the structure of totally global functors in $\mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ when k is algebraically closed.

Throughout this section, we let k be an algebraically closed field, we assume X and Y are noetherian, and we let

$$\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$$

denote the category of k-linear functors from $\mathsf{Qcoh}X$ to $\mathsf{Qcoh}Y$ which take coherent objects to coherent objects. If F is an object of $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$, we let $F|_{\mathsf{coh}X}$ denote the restriction of F to the full subcategory of $\mathsf{Qcoh}X$ consisting of coherent objects.

In order to simplify the exposition, we introduce the concept of an *admissible* functor.

Definition 7.1. Suppose X is a projective variety with very ample invertible sheaf $\mathcal{O}(1)$. A nonzero object F in $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is called an *admissible functor* if it

(1) is totally global
(2) is half-exact on vector-bundles,
(3) commutes with direct limits, and
(4) has the property that $F\alpha$ is epic for all nonzero $\alpha \in \mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(n))$.

For $i \in \mathbb{Z}$, the functor $H^1(\mathbb{P}^1, (-)(i))$ is admissible.

Our main result in this section (Proposition 7.6, Corollary 7.10) is that an admissible functor $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ admits a split monic

$$\Delta : H^1(\mathbb{P}^1, (-)(i)) \longrightarrow F$$

for some $i \in \mathbb{Z}$. This allows us to prove (Theorem 7.12) that every admissible functor in $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is a direct sum of cohomologies. Since a nonzero, totally global functor $F \in \mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ is admissible (Corollary 7.3), the same holds for such functors.

Lemma 7.2. *Let X be a projective variety with very ample invertible sheaf $\mathcal{O}(1)$, and suppose $F \in \mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$ is right exact and vanishes on coherent torsion modules. Then F satisfies (4) in Definition 7.1.*

Proof. If $\alpha \in \mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ and either $F\mathcal{O}(n) = 0$ or $m > n$, then $\alpha = 0$. If $m = n$, and α is not zero, them α is an isomorphism so that $F\alpha$ is epic. Thus, suppose $F\mathcal{O}(n) \neq 0$, let $m < n$ and let $\alpha \in \mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ be nonzero. We first

show that the kernel of α must be zero. If not, pick an affine open cover over which both $\mathcal{O}(m)$ and $\mathcal{O}(n)$ are free. Over one of these sets, U, $\ker\alpha$ is nonzero. Since $\alpha(U)$ is just multiplication by some element of $\mathcal{O}(U)$, and since X is integral, $\alpha(U)$ must be the zero map. Therefore, $U \subset \operatorname{Supp}\ker\alpha$. On the other hand, since $\ker\alpha$ is coherent, its support is closed in X. Since X is integral, the support of $\ker\alpha$ must equal X. But the support of $\ker\alpha$ is disjoint from the set of points $p \in X$ such that $\alpha_p \neq 0$, since this map is just multiplication by a nonzero element of a domain. We conclude that the kernel of α equals 0.

The cokernel of α_p is a torsion $\mathcal{O}_{X,p}$-module for all p. We conclude that the cokernel of α is torsion. Therefore, there is an exact sequence

$$0 \to \mathcal{O}(m) \xrightarrow{\alpha} \mathcal{O}(n) \to \mathcal{T} \to 0$$

with $\mathcal{T}$ torsion. Hence $\dim F\mathcal{O}(m) \geq \dim F\mathcal{O}(n)$ by the right exactness of F and by the fact that $F\mathcal{T} = 0$. □

Corollary 7.3. *If $F \in \mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ is non-zero and totally global, then F is admissible.*

Proof. Since $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is totally global, F vanishes on coherent torsion modules by Lemma 4.2. Therefore, F is admissible by Lemma 7.2. □

7.1. **Subfunctors of Admissible Functors.** In this subsection we prove that if $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible, it has a subfunctor isomorphic to $H^1(\mathbb{P}^1, (-)(i))$ for some integer i. We begin with some preliminary results.

Lemma 7.4. *Let X be a projective variety with very ample invertible sheaf $\mathcal{O}(1)$ such that for all $i > 0$, we have*

$$\dim_k \Gamma(X, \mathcal{O}(i)) > 1.$$

If $F \in \mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$ satisfies (4) in Definition 7.1 and $F\mathcal{O}(n) \neq 0$ for some $n \in \mathbb{Z}$, then

$$\dim_k F\mathcal{O}(m) > \dim_k F\mathcal{O}(n)$$

for all $m < n$.

Proof. Let n be such that $F\mathcal{O}(n) \neq 0$ and suppose that for all nonzero $\alpha \in \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ with $m < n$ we have $F\alpha$ epic. To prove the assertion, we must exclude the possibility that there exists some $m < n$ such that $\dim_k F\mathcal{O}(m) = \dim_k F\mathcal{O}(n)$. Suppose to the contrary that for some $m < n$, $\dim_k F\mathcal{O}(m) = d = \dim_k F\mathcal{O}(n) \neq 0$. Then, for all nonzero $\alpha \in \operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$, $F\alpha$ is an isomorphism. Pick a basis $\alpha_0, \ldots, \alpha_r$ for $\operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n))$ and let $x_0, \ldots, x_r$ denote indeterminates. Note that by hypothesis, $r > 0$. Since

$$\det(x_0 F\alpha_0 + \cdots + x_r F\alpha_r)$$

is a homogeneous polynomial of degree $d > 0$ in $k[x_0, \ldots, x_r]$, it has a non-trivial zero which then gives a non-zero α such that $F\alpha$ is not invertible. This is a contradiction. □

The following lemma will be invoked in the proof of Proposition 7.6. Its straightforward proof is omitted.

Lemma 7.5. *Suppose $F_1, F_2 \in \mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ preserve direct limits.*

If $\underline{\Delta} : F_1|_{\mathsf{coh}X} \longrightarrow F_2|_{\mathsf{coh}X}$ is a natural transformation, then $\underline{\Delta}$ extends uniquely to a natural transformation $\Delta : F_1 \longrightarrow F_2$. If $\underline{\Delta}$ is monic, i.e. if $\underline{\Delta}_{\mathcal{M}}$ is monic for all coherent objects $\mathcal{M}$ in $\mathsf{Qcoh}X$, then Δ is monic in $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$. If $\underline{\Delta}$ is epic, then Δ is epic in $\mathsf{Funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$.

We introduce notation which will be used in the proof of Proposition 7.6: let $A = k[x_0, x_1]$ denote the polynomial ring in 2 variables with its usual grading, let $[-]$ denote the shift functor, and let $f_i : A[-(n+1)] \to A[-n]$ and $g_i : A[-(n+2)] \to A[-(n+1)]$ denote multiplication by x_i. Then we have a short exact sequence in $\mathsf{Gr}A$:

$$0 \longrightarrow A[-(n+2)] \stackrel{(g_1,-g_0)}{\longrightarrow} A[-(n+1)]^{\oplus 2} \stackrel{f_0+f_1}{\longrightarrow} A[-n] \longrightarrow k[-n] \longrightarrow 0$$

where k denotes the trivial module. This induces the short exact sequence

$$0 \longrightarrow \mathcal{O}(-(n+2)) \stackrel{(\phi_1,-\phi_0)}{\longrightarrow} \mathcal{O}(-(n+1))^{\oplus 2} \stackrel{\psi_0+\psi_1}{\longrightarrow} \mathcal{O}(-n) \longrightarrow 0. \tag{64}$$

Proposition 7.6. *Suppose $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible. Then the set*

$$\{i \in \mathbb{Z} | F\mathcal{O}(i) \neq 0\}$$

has a maximum, r, and there is a monic morphism

$$\Delta : H^1(\mathbb{P}^1, (-)(-2-r)) \to F$$

in $\mathsf{Funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$.

Proof. We first show that r is well defined. Since F is non-zero and totally global, $F\mathcal{O}(n) \neq 0$ for some n. Then $\dim F\mathcal{O}(n) > \dim F\mathcal{O}(n+1)$ by Lemma 7.4, so $F\mathcal{O}(i) = 0$ for all $i >> 0$. Hence, the set $\{i \in \mathbb{Z} | F\mathcal{O}(i) \neq 0\}$ indeed has a maximum.

We let $H := H^1(\mathbb{P}^1, (-)(-2-r))$ and note that $H\mathcal{O}(r) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong k$. We first define a natural transformation $\underline{\Delta} : H|_{\mathsf{coh}\mathbb{P}^1} \to F|_{\mathsf{coh}\mathbb{P}^1}$ by defining $\underline{\Delta}_{\mathcal{F}}$ for each indecomposable coherent sheaf $\mathcal{F}$. If $\mathcal{F}$ is torsion or isomorphic to $\mathcal{O}(i)$ with $i > r$ we define $\underline{\Delta}_{\mathcal{F}} = 0$, and we define $\underline{\Delta}_{\mathcal{O}(r)} : H\mathcal{O}(r) \to F\mathcal{O}(r)$ to be any nonzero map. Now suppose we have defined $\underline{\Delta}_{\mathcal{O}(i)}$ for all $i > m$ such that each such $\underline{\Delta}_{\mathcal{O}(i)}$ is injective and such that

$$\begin{array}{ccc} H\mathcal{O}(j) & \xrightarrow{H\psi} & H\mathcal{O}(j+1) \\ {\scriptstyle \underline{\Delta}_{\mathcal{O}(j)}}\downarrow & & \downarrow{\scriptstyle \underline{\Delta}_{\mathcal{O}(j+1)}} \\ F\mathcal{O}(j) & \xrightarrow{F\psi} & F\mathcal{O}(j+1) \end{array}$$

commutes for $j \geq i$ and $\psi \in \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(j), \mathcal{O}(j+1))$. We construct an injective homomorphism $\theta : H\mathcal{O}(m) \to F\mathcal{O}(m)$ such that

$$\begin{array}{ccc} H\mathcal{O}(m) & \xrightarrow{H\phi} & H\mathcal{O}(m+1) \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \underline{\Delta}_{\mathcal{O}(m+1)}} \\ F\mathcal{O}(m) & \xrightarrow{F\phi} & F\mathcal{O}(m+1) \end{array} \tag{65}$$

commutes for $\phi = \phi_0, \phi_1$ (see (64) for a definition of these maps). To this end, we apply both H and F to the exact sequence (64) with $n := -m-2$ to get a diagram

$$\begin{array}{ccccc} H\mathcal{O}(m) & \stackrel{(H\phi_1,-H\phi_0)}{\longrightarrow} & H\mathcal{O}(m+1)^{\oplus 2} & \stackrel{H\psi_0+H\psi_1}{\longrightarrow} & H\mathcal{O}(m+2) \\ & & \downarrow \underline{\Delta}^{\oplus 2}_{\mathcal{O}(m+1)} & & \downarrow \underline{\Delta}_{\mathcal{O}(m+2)} \\ F\mathcal{O}(m) & \stackrel{(F\phi_1,-F\phi_0)}{\longrightarrow} & F\mathcal{O}(m+1)^{\oplus 2} & \stackrel{F\psi_0+F\psi_1}{\longrightarrow} & F\mathcal{O}(m+2) \end{array} \tag{66}$$

with exact rows whose right square commutes.

To construct θ, choose a basis $u_1, \dots, u_{r-m+1}$ for $H\mathcal{O}(m)$. Now,

$$(H\phi_1, -H\phi_0)(u_i) \in \ker(H\psi_0 + H\psi_1).$$

Thus, by the commutativity of the right-hand square of (66),

$$(\underline{\Delta}_{\mathcal{O}(m+1)}, \underline{\Delta}_{\mathcal{O}(m+1)})(H\phi_1(u_i), -H\phi_0(u_i))$$

is in the image of $(F\phi_1, -F\phi_0)$. Hence, there exists a $v_i \in F\mathcal{O}(m)$ such that $F\phi_j(v_i) = \underline{\Delta}_{\mathcal{O}(m+1)} H\phi_j(u_i)$ for $i = 1, \dots, r-m+1$ and $j = 0, 1$. We define $\theta(u_i) = v_i$. Since F is k-linear, we conclude that (65) commutes for all $\phi \in \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}(m+1))$. We define $\underline{\Delta}_{\mathcal{O}(m)} := \theta$, and we note that $\underline{\Delta}_{\mathcal{O}(m)}$ is monic since $(H\phi_1, -H\phi_0)$ is monic.

Next, we define $\underline{\Delta}_{\mathcal{F}}$ when $\mathcal{F}$ is isomorphic to $\mathcal{O}(n)$. Let $\alpha : \mathcal{F} \to \mathcal{O}(n)$ be an isomorphism. Define

$$\underline{\Delta}_{\mathcal{F}} := (F\alpha)^{-1} \circ \underline{\Delta}_{\mathcal{O}(n)} \circ H\alpha.$$

If $\beta : \mathcal{F} \to \mathcal{O}(n)$ is another isomorphism, then $\beta = \lambda\alpha$ for some $0 \neq \lambda \in k$, whence $(F\beta)^{-1} = \lambda^{-1}(F\alpha)^{-1}$ and $H\beta = \lambda H\alpha$; thus the definition of $\delta_{\mathcal{F}}$ does not depend on the choice of α.

We now define $\underline{\Delta}_{\mathcal{F}}$ for arbitrary $\mathcal{F}$ by writing $\mathcal{F}$ as a direct sum of indecomposables, say $\mathcal{F} = \oplus \mathcal{F}_i$, and defining $\underline{\Delta}_{\mathcal{F}} := \oplus \underline{\Delta}_{\mathcal{F}_i}$.

To show that $\underline{\Delta}$ is a natural transformation we must show that

$$\begin{array}{ccc} H\mathcal{F} & \stackrel{Hf}{\longrightarrow} & H\mathcal{G} \\ \underline{\Delta}_{\mathcal{F}} \downarrow & & \downarrow \underline{\Delta}_{\mathcal{G}} \\ F\mathcal{F} & \underset{Ff}{\longrightarrow} & F\mathcal{G} \end{array} \tag{67}$$

commutes for all $\mathcal{F}$ and $\mathcal{G}$ and all maps $f : \mathcal{F} \to \mathcal{G}$. It suffices to check this when $\mathcal{F}$ and $\mathcal{G}$ are indecomposable. The diagram commutes when $\mathcal{G}$ is torsion because $F\mathcal{G} = 0$ then. If $\mathcal{G}$ is torsion-free and $\mathcal{F}$ torsion, then $f = 0$ so the diagram commutes. Thus, the only remaining case is that when $\mathcal{F} \cong \mathcal{O}(i)$ and $\mathcal{G} \cong \mathcal{O}(j)$ with $i \leq j$. The case $i = j$ is straightforward and we omit the verification in this case. Thus, we may suppose $i > j$.

Write $f = \beta^{-1} g \alpha$ were $\alpha : \mathcal{F} \to \mathcal{O}(i)$ and $\beta : \mathcal{G} \to \mathcal{O}(j)$ are isomorphisms and $0 \neq g : \mathcal{O}(i) \to \mathcal{O}(j)$. We can write g as a sum of terms of the form $\psi_j \psi_{j-1} \cdots \psi_{i+1}$ where each $\psi_l : \mathcal{O}(l-1) \to \mathcal{O}(l)$ is monic. Now

$$\underline{\Delta}_{\mathcal{O}(j)} \circ H\psi_j \circ \cdots \circ H\psi_{i+1} = F\psi_j \circ \cdots \circ F\psi_{i+1} \circ \underline{\Delta}_{\mathcal{O}(i)}$$

and this implies

$$\underline{\Delta}_{\mathcal{O}(j)} \circ Hg = Fg \circ \underline{\Delta}_{\mathcal{O}(i)}.$$

Therefore,

$$\begin{aligned}\underline{\Delta}_{\mathcal{G}} \circ Hf &= F\beta^{-1} \circ \underline{\Delta}_{\mathcal{O}(j)} \circ H\beta \circ Hf \\ &= F\beta^{-1} \circ \underline{\Delta}_{\mathcal{O}(j)} \circ Hg \circ H\alpha \\ &= F\beta^{-1} \circ Fg \circ \underline{\Delta}_{\mathcal{O}(i)} \circ H\alpha \\ &= Ff \circ F\alpha^{-1} \circ \underline{\Delta}_{\mathcal{O}(i)} \circ H\alpha \\ &= Ff \circ \underline{\Delta}_{\mathcal{F}}.\end{aligned}$$

This shows that (67) commutes and so completes the proof that Δ is natural.

Finally, $\underline{\Delta}_{\mathcal{F}}$ is monic for all indecomposable coherent $\mathcal{F}$ and hence for all coherent $\mathcal{F}$. It follows from Lemma 7.5 that $\underline{\Delta}$ extends to a monic natural transformation

$$\Delta : H \longrightarrow F.$$

□

7.2. **The Structure of Admissible Functors in $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$.** In this subsection, we work towards a proof, realized in Corollary 7.10, that the monic Δ constructed in Proposition 7.6 is split. It follows (Theorem 7.12) that an admissible functor in $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is a direct sum of cohomologies. We assume, throughout the subsection, that X and Y are projective schemes, $\mathcal{F}, \mathcal{G}, \mathcal{M} \in \mathsf{Qcoh}X$ are coherent, and F is an object of $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$.

We first define a natural transformation

$$\Phi_F : F|_{\mathsf{coh}X} \longrightarrow \mathrm{Hom}(-, \mathcal{G})^*|_{\mathsf{coh}X} \otimes_k F\mathcal{G}$$

which will be used to split the monic Δ constructed in Proposition 7.6. To this end, we let

$$\eta_{\mathcal{F},\mathcal{G}} : k \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$$

be defined as follows: $\eta_{\mathcal{F},\mathcal{G}}(a) := a(\sum_i f_i^* \otimes f_i)$ where $\{f_1, \ldots, f_m\}$ is a basis for $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$. We next note that the functor F induces a map

$$\phi_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) \otimes_k F\mathcal{F} \longrightarrow F\mathcal{G} \tag{68}$$

as follows: if U is an open set in Y, and $s \in F\mathcal{F}(U)$, we define (68) over U to be the map

$$f \otimes s \mapsto F(f)(U)(s).$$

We define the natural transformation

$$\Phi_F : F|_{\mathsf{coh}X} \longrightarrow \mathrm{Hom}(-, \mathcal{G})^*|_{\mathsf{coh}X} \otimes_k F\mathcal{G}. \tag{69}$$

as follows:

$$\Phi_{F\mathcal{F}} : F\mathcal{F} \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k F\mathcal{G}$$

is defined to be the composition of

$$\eta_{\mathcal{F},\mathcal{G}} \otimes_k F\mathcal{F} : F\mathcal{F} \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \otimes_k F\mathcal{F}$$

with

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \phi_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}) \otimes_k F\mathcal{F} \longrightarrow$$

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k F\mathcal{G}.$$

The proof that Φ_F is natural is straightforward and we omit it.

Lemma 7.7. *If $\mathcal{N}$ is a coherent object of $\mathsf{Qcoh}Y$, $\mathcal{G}$ is an invertible $\mathcal{O}_X$-module and*

$$F = \mathrm{Hom}_X(-, \mathcal{G})^* \otimes_k \mathcal{N},$$

then the morphism Φ_F is an isomorphism.

Proof. Let $\{f_1, \dots, f_m\}$ be a basis for $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ and let U be open in Y. Then

$$\Phi_{F\mathcal{F}}(U) : F\mathcal{F}(U) \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k F\mathcal{G}(U)$$

sends $s \in F\mathcal{F}(U)$ to $\sum_{i=1}^m f_i^* \otimes F(f_i)(U)(s)$.

Suppose s is a simple tensor, so

$$s = \delta \otimes t \in \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \mathcal{N}(U).$$

We describe $F(f_i)(U)(\delta \otimes t)$. The map $f_i : \mathcal{F} \longrightarrow \mathcal{G}$ induces the map

$$- \circ f_i : \mathrm{Hom}_X(\mathcal{G}, \mathcal{G}) \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{G}).$$

Dualizing gives a map

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{G})^*$$

which sends δ to $\delta \circ (- \circ f_i)$. Therefore, $F(f_i)(U)(\delta \otimes t) = \delta \circ (- \circ f_i) \otimes t$ and so the morphism

$$\Phi_{F\mathcal{F}}(U) : \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \mathcal{N}(U) \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{G}, \mathcal{G})^* \otimes_k \mathcal{N}(U)$$

sends $\delta \otimes t$ to $\sum_i f_i^* \otimes (\delta \circ (- \circ f_i)) \otimes t$. Since the map $\delta \circ (- \circ f_i) \in \mathrm{Hom}_X(\mathcal{G}, \mathcal{G})^* \cong k$ sends multiplication by α to multiplication by $\alpha\delta(f_i)$, the function

$$\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{G}, \mathcal{G})^*$$

defined by sending δ to $\sum_i f_i^* \otimes_k (\delta \circ (- \circ f_i))$ is injective and k-linear, hence an isomorphism of vector spaces. It follows that $\Phi_{F\mathcal{F}}(U)$ is a tensor product of two isomorphisms, and the assertion follows. □

Lemma 7.8. *Let $\Theta : F' \longrightarrow F$ be a natural transformation between elements of $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$. Then the diagram*

$$\begin{array}{ccc} F & \xrightarrow{\Phi_F} & \mathrm{Hom}_X(-, \mathcal{G})^* \otimes_k F\mathcal{G} \\ {\scriptstyle\Theta}\uparrow & & \uparrow \\ F' & \xrightarrow[\Phi_{F'}]{} & \mathrm{Hom}_X(-, \mathcal{G})^* \otimes_k F'\mathcal{G} \end{array}$$

whose right vertical is induced by Θ, commutes on coherent objects.

Proof. From the definition of Φ, it suffices to show that the diagram

$$\begin{array}{ccc} F\mathcal{M} & \longrightarrow & \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{M}, \mathcal{G}) \otimes_k F \\ {\scriptstyle\Theta_{\mathcal{M}}}\uparrow & & \uparrow \\ F'\mathcal{M} & \longrightarrow & \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{M}, \mathcal{G}) \otimes_k F' \end{array} \tag{70}$$

whose right vertical is induced by Θ and whose horizontals are induced by the unit map $k \longrightarrow \mathrm{Hom}_X(-, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(-, \mathcal{G})$ commutes, and that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_X(\mathcal{M}, \mathcal{G}) \otimes_k F\mathcal{M} & \longrightarrow & F\mathcal{G} \\ \uparrow & & {\scriptstyle\Theta_{\mathcal{G}}}\uparrow \\ \mathrm{Hom}_X(\mathcal{M}, \mathcal{G}) \otimes_k F'\mathcal{M} & \longrightarrow & F'\mathcal{G} \end{array} \tag{71}$$

whose left vertical is induced by Θ and whose horizontals are induced by evaluation, commutes. The fact that (70) commutes is trivial. We check commutativity of (71). The top route of (71) evaluated on the open set $U \subset Y$ sends $f \otimes x$ to $F(f)(U)(\Theta_{\mathcal{M}}(U)(x))$ while the bottom route of (71) sends $f \otimes x$ to $\Theta_{\mathcal{G}}(U)F'(f)(U)(x)$. These values are equal by the naturality of Θ. □

Lemma 7.9. *If $F \in \mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$ is such that there exists an invertible $\mathcal{G} \in \mathsf{Qcoh}X$ and a monomorphism*

$$\Psi : \mathrm{Hom}_X(-, \mathcal{G})^* \longrightarrow F$$

in $\mathsf{funct}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}\mathbb{P}^0)$, then the restriction of Ψ to coherents,

$$\underline{\Psi} : \mathrm{Hom}_X(-, \mathcal{G})^*|_{\mathsf{coh}X} \longrightarrow F|_{\mathsf{coh}X},$$

splits.

Proof. Let $\psi : F\mathcal{G} \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{G})^*$ be a splitting of $\Psi_{\mathcal{G}}$. Consider the diagram

$$\begin{array}{ccc} F\mathcal{M} & \xrightarrow{\Phi_{F\mathcal{M}}} & \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* \otimes_k F\mathcal{G} \\ {\scriptstyle \Psi_{\mathcal{M}}}\uparrow & & \uparrow \\ \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* & \xrightarrow[\Phi_{\mathrm{Hom}_X(-,\mathcal{G})^*\mathcal{M}}]{} & \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{G}, \mathcal{G})^* \end{array}$$

whose right vertical is induced by $\Psi_{\mathcal{G}}$. The bottom horizontal is an isomorphism by Lemma 7.7, and the diagram commutes by Lemma 7.8. It follows that the diagram

$$\begin{array}{ccc} F\mathcal{M} & \xrightarrow{\Phi_{F\mathcal{M}}} & \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* \otimes_k F\mathcal{G} \\ {\scriptstyle \Psi_{\mathcal{M}}}\uparrow & & \downarrow \\ \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* & \xrightarrow[\Phi_{\mathrm{Hom}_X(-,\mathcal{G})^*\mathcal{M}}]{} & \mathrm{Hom}_X(\mathcal{M}, \mathcal{G})^* \otimes_k \mathrm{Hom}_X(\mathcal{G}, \mathcal{G})^* \end{array}$$

whose right vertical is induced by ψ, commutes. The lemma follows. □

Corollary 7.10. *Let $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ be admissible. The monic*

$$\Delta : H^1(\mathbb{P}^1, (-)(-2-r)) \longrightarrow F$$

constructed in Proposition 7.6 splits.

Proof. The monic Δ restricts to a monic

$$\underline{\Delta} : H^1(\mathbb{P}^1, (-)(-2-r))|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow F|_{\mathsf{coh}\mathbb{P}^1}.$$

By Serre duality, $\underline{\Delta}$ induces a monic

$$\underline{\Delta}' : \mathrm{Hom}_{\mathbb{P}^1}(-, \mathcal{O}(r))^*|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow F|_{\mathsf{coh}\mathbb{P}^1}$$

which by Lemma 7.9, admits a splitting

$$\underline{\Psi}' : F|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow \mathrm{Hom}_{\mathbb{P}^1}(-, \mathcal{O}(r))^*|_{\mathsf{coh}\mathbb{P}^1}.$$

The map $\underline{\Psi}'$ induces, by Serre duality again, a splitting

$$\underline{\Psi} : F|_{\mathsf{coh}\mathbb{P}^1} \longrightarrow H^1(\mathbb{P}^1, (-)(-2-r))|_{\mathsf{coh}\mathbb{P}^1}$$

of $\underline{\Delta}$. We claim that $\underline{\Psi}$ extends to a splitting

$$\Psi : F \longrightarrow H^1(\mathbb{P}^1, (-)(-2-r))$$

of Δ. To this end, Lemma 7.5 implies that $\underline{\Psi}$ has a unique extension Ψ. We also know that $\Psi\Delta$ restricts on coherent objects to the map $\underline{\Psi}\ \underline{\Delta} = \mathrm{id}_{H^1(\mathbb{P}^1,(-)(-2-r))|_{\mathsf{coh}\mathbb{P}^1}}$. But by Lemma 7.5, $\underline{\Psi}\ \underline{\Delta}$ extends uniquely to a natural transformation

$$H^1(\mathbb{P}^1,(-)(-2-r)) \longrightarrow H^1(\mathbb{P}^1,(-)(-2-r)).$$

Thus, $\Psi\Delta = \mathrm{id}_{H^1(\mathbb{P}^1,(-)(-2-r))}$, whence the Corollary. □

We omit the straightforward proof of the following

Lemma 7.11. *Suppose $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible.*

If $F \cong A \oplus B$ in $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$, and if A is non-zero, then A is admissible as well.

Theorem 7.12. *If $F \in \mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$ is admissible, then there exist integers $m, n_i \geq 0$ such that*

$$F \cong \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1,(-)(i))^{\oplus n_i}.$$

Proof. Since F is admissible, Proposition 7.6 implies that the set $\{i|F\mathcal{O}(i) \neq 0\}$ has a maximum, r. We let $m = r + 2$. Since F preserves coherence, the set

$$\{n|\text{there exists a split monomorphism } H^1(\mathbb{P}^1,(-)(-m))^{\oplus n} \to F\}$$

has a maximum, which we call n_0. If we let $F_0 = H^1(\mathbb{P}^1,(-)(-m))^{\oplus n_0}$, and we let $\delta_0 : F_0 \to F$ be a split monomorphism, then there is a sub-functor $F^{(1)}$ of F such that $F \cong F_0 \oplus F^{(1)}$. By Lemma 7.11, either $F^{(1)}$ is 0 or $F^{(1)}$ is admissible.

Now, given a sub-functor $F^{(i)}$ of F which is either 0 or admissible, we construct an object F_i in the category $\mathsf{funct}_k(\mathsf{Qcoh}\mathbb{P}^1, \mathsf{Qcoh}\mathbb{P}^0)$, a split monomorphism $\delta_i : F_i \to F^{(i)}$, and a sub-functor $F^{(i+1)}$ of $F^{(i)}$ which is either 0 or admissible, as follows. We let

$$n_i = \max\{n|\text{there exists a split monomorphism } H^1(\mathbb{P}^1,(-)(-m+i))^{\oplus n} \to F^{(i)}\},$$

we let $F_i = H^1(\mathbb{P}^1,(-)(-m+i))^{\oplus n_i}$, and we let $\delta_i : F_i \to F^{(i)}$ be a split monomorphism. Then there is a sub-functor $F^{(i+1)}$ of $F^{(i)}$ such that $F^{(i)} \cong F_i \oplus F^{(i+1)}$. By Lemma 7.11, either $F^{(i+1)}$ is 0, or $F^{(i+1)}$ is admissible.

In this way we get a morphism

$$\Delta : \oplus_{i=0}^{\infty} H^1(\mathbb{P}^1,(-)(-m+i))^{\oplus n_i} \to F.$$

defined by $\Delta := \oplus_{i=0}^{\infty}\delta_i$. We claim that Δ is an isomorphism. By Lemma 7.5, it suffices to show that $\Delta|_{\mathsf{coh}\mathbb{P}^1}$ is an isomorphism. To this end, let $\mathcal{M}$ be a coherent $\mathcal{O}_{\mathbb{P}^1}$-module. Then $\mathcal{M} \cong \mathcal{O}(i_1) \oplus \cdots \oplus \mathcal{O}(i_n) \oplus \mathcal{T}$ where $\mathcal{T}$ is coherent torsion and $i_1 = \min\{i_1, \ldots, i_n\}$. It follows that

$$\oplus_{i=0}^{\infty} H^1(\mathbb{P}^1,\mathcal{M}(-m+i))^{\oplus n_i} = \oplus_{i=0}^{-2-i_1+m} H^1(\mathbb{P}^1,\mathcal{M}(-m+i))^{\oplus n_i}.$$

By the construction of Δ, in order to show that $\Delta_{\mathcal{M}}$ is an isomorphism, it suffices to show that $F^{(-1-i_1+m)}(\mathcal{M}) = 0$. If not, then $F^{(-1-i_1+m)}$ is an admissible direct summand of F. By Proposition 7.6, the set $\{i|F^{(-1-i_1+m)}\mathcal{O}(i) \neq 0\}$ has a maximum, s, and there exists a split monomorphism $H^1(\mathbb{P}^1,(-)(-2-s)) \to F^{(-1-i_1+m)}$. Since $F^{(-1-i_1+m)}$ is totally global and $F^{(-1-i_1+m)}(\mathcal{M})$ is nonzero, it follows that one of

$$F^{(-1-i_1+m)}(\mathcal{O}(i_1)), \ldots, F^{(-1-i_1+m)}(\mathcal{O}(i_n))$$

is nonzero. Hence $i_1 \leq s$. Since $m = r + 2$ and $s \leq r$, it follows that $-m \leq -s - 2$. Thus, we have

$$-m \leq -2 - s \leq -2 - i_1.$$

This contradicts the maximality of n_{-2-s+m}. □

By Corollary 7.3, Theorem 7.12 immediately implies the following

Corollary 7.13. *If $F \in \mathsf{bimod}_k(\mathbb{P}^1 - \mathbb{P}^0)$ is totally global, then F is a direct sum of cohomologies, i.e. there exist integers $m, n_i \geq 0$ such that*

$$F \cong \oplus_{i=-m}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}.$$

References

[1] S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models*, Springer-Verlag, 1990.
[2] B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Math. 1750, Springer-Verlag, 2000.
[3] S. Eilenberg, Abstract description of some basic functors, *J. Indian Math. Soc.(N.S.),* **24** (1960) 231-234.
[4] R. Harthshorne, *Algebraic geometry*, Springer-Verlag, 1977.
[5] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. 20, Springer-Verlag, 1966.
[6] C. Ingalls and D. Patrick, *Blowing up quantum weighted projective planes*, J. Algebra 254, (2002) 92–114.
[7] A. Nyman and S. P. Smith, *A generalization of Watts's Theorem: right exact functors on module categories*, preprint.
[8] A. Polishchuk, *Kernel algebras and generalized Fourier-Mukai transforms*, preprint.
[9] M. Van den Bergh, Blowing up of non-commutative smooth surfaces, *Mem. Amer. Math. Soc.*, **154** (2001).
[10] M. Van den Bergh, *Non-commutative $\mathbb{P}^1$-bundles over commutative schemes*, preprint.
[11] C. E. Watts, Intrinsic characterizations of some additive functors, *Proc. Amer. Math. Soc.,* **11** (1960) 5-8.

Department of Mathematics, 516 High St, Western Washington University, Bellingham, WA 98225-9063

E-mail address: adam.nyman@wwu.edu