The Geometry of (Some) Noncommutative Projective Lines

Adam Nyman

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Conventions and Notation

- $k$ a perfect field
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- $L/k$ finite extension
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- $k$ a perfect field
- $L/k$ finite extension
- $\overline{L}$ an algebraic closure of $L$
Part 1

Noncommutative Projective Lines
Noncommutative Space := Grothendieck Category
Noncommutative Space $\equiv$ Grothendieck Category $=$

- $(k$-linear) abelian category with
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- (\(k\)-linear) abelian category with
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- \text{Mod } R, R \text{ a ring}
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Examples

- Mod $R$, $R$ a ring
- Qcoh $X$
Noncommutative Spaces

Noncommutative Space :\(=\) Grothendieck Category =

- (\(k\)-linear) abelian category with
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- a generator.

**Examples**

- \(\text{Mod } R, R\) a ring
- \(\text{Qcoh } X\)
- \(\text{Proj } A :\! = \text{Gr}A/\text{Tors}A\) where \(A\) is \(\mathbb{Z}\)-graded
(Commutative) polynomial ring $k[x_1, \ldots , x_n]$ has $\mathbb{Z}^n$-grading:
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$$|x_i| = (0, \ldots, 0, 1, 0, \ldots, 0).$$
Curves on Quasischemes (Smith and Zhang (1998))

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The noncommutative space $\mathbb{V}_n^1$
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If $X$ is noncommutative space, $Y$ is a regularly embedded hypersurface, and $C$ is a curve which is ‘in good position’ w.r.t. $Y$, then
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Significance

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Examples
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**Examples**

1. $\text{coh}\mathbb{P}^1$
2. Weighted projective lines (Geigle-Lenzing)
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Examples

1. \( \text{coh}\mathbb{P}^1 \)
2. Weighted projective lines (Geigle-Lenzing)
3. Arithmetic noncommutative projective lines
Spaces of form $\text{Proj}S^{n.c.}(V) =: \mathbb{P}^{n.c.}(V)$ where
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- $V$ is a two-sided vector space
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- $\text{Proj} A = \text{Gr} A / \text{Tors} A$. 
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Theme of talk
Spaces of form $\text{Proj} S^{n.c.}(V) =: \mathbb{P}^{n.c.}(V)$ where
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Theme of talk

Study $V \mapsto \mathbb{P}^{n.c.}(V)$
Spaces of form $\text{Proj} S^{n.c.}(V) =: \mathbb{P}^{n.c.}(V)$ where

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**Theme of talk**

\[ \text{Study } V \rightsquigarrow \mathbb{P}^{n.c.}(V) \]

Initial Motivation: The noncommutative geometry of $\mathbb{P}^{n.c.}(V)$ is well understood.
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**Remark**

The classification of noncommutative curves due to Reiten and Van den Bergh (2002) is over $k = \overline{k}$. 

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**Remark**

The classification of noncommutative curves due to Reiten and Vanden Bergh (2002) is over $k = \overline{k}$. In this case $\mathbb{P}^{n.c.}(V) \equiv \text{Qcoh}\mathbb{P}^1$. 
Part 2

Two-sided Vector Spaces
Basic Terminology
A two-sided vector space of rank $n$ is a
A **two-sided vector space of rank** \( n \) is a

- **\( k \)-central \( L-L \)-bimodule** \( V \) such that
A two-sided vector space of rank $n$ is a

- $k$-central $L$-$L$-bimodule $V$ such that
- $\dim_L(LV) = \dim_L(V_L) = n$. 
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$k = \mathbb{R}, \quad L = \mathbb{C}, \quad V = \mathbb{C}, \quad \sigma = \text{complex conjugation}$
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$k = \mathbb{R}, \ L = \mathbb{C}, \ V = \mathbb{C}, \ \sigma = \text{complex conjugation} \ x \cdot v := xv$
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$v \cdot x := v \sigma(x)$
A **two-sided vector space of rank** $n$ is a

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**Example 2**
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**Example 2**

$V = L^n$, $\phi : L \to M_n(L)$
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$v \cdot x := v\sigma(x)$ Notation: $\mathbb{C}_\sigma$

**Example 2**

$V = L^n$, $\phi : L \rightarrow M_n(L)$ $x \cdot v = xv$ $v \cdot x = v\phi(x)$ Notation: $L^n_\phi$
Theorem (Patrick 2000)

Suppose \( \text{char } k \neq 2 \). If \( V \) has rank 2, either
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Suppose char \( k \neq 2 \). If \( V \) has rank 2, either

1. \( V \cong L_2^\phi \) where \( \phi(x) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & \sigma(x) \end{pmatrix} \) where \( \sigma(x) \in \text{Gal}(L/k) \),
Theorem (Patrick 2000)

Suppose $\text{char } k \neq 2$. If $V$ has rank 2, either

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2. $V \cong L_2^\phi$ where $\phi(x) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & \tau(x) \end{pmatrix}$, $\sigma(x), \tau(x) \in \text{Gal}(L/k)$, and $\tau \neq \sigma$, or
Classification of Rank 2 Two-sided Vector Spaces

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2. $V \cong L_\phi^2$ where $\phi(x) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & \tau(x) \end{pmatrix}$, $\sigma(x), \tau(x) \in \text{Gal}(L/k)$, and $\tau \neq \sigma$, or

3. $V$ is simple.
Theorem (Patrick 2000)

Suppose \( \text{char } k \neq 2 \). If \( V \) has rank 2, either

1. \( V \cong L^2_\phi \) where \( \phi(x) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & \sigma(x) \end{pmatrix} \) where \( \sigma(x) \in \text{Gal}(L/k) \),

2. \( V \cong L^2_\phi \) where \( \phi(x) = \begin{pmatrix} \sigma(x) & 0 \\ 0 & \tau(x) \end{pmatrix} \), \( \sigma(x), \tau(x) \in \text{Gal}(L/k) \), and \( \tau \neq \sigma \), or

3. \( V \) is simple. In this case \( V \cong L^2_\phi \) where

\[
\phi(x) = \begin{pmatrix} a(x) & b(x) \\ mb(x) & a(x) \end{pmatrix}
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Simple Two-sided Vector Spaces I: Classification

- $\text{Emb}(L) = \{ k - \text{linear embeddings } L \to \overline{L} \}$
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There is a bijection

$$\Phi : \text{Orb}(L) \to \text{Simp}(L)$$
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Notation: \( \Phi(\lambda^G) = [V(\lambda)]. \)
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**Remark**

The result holds even if \( L/k \) is infinite
Construction of $V(\lambda)$
What is $V(\lambda)$?
What is $V(\lambda)$?

$V(\lambda) := 1L \lor \lambda(L)_{\lambda}$

Action defined as $a \cdot v \cdot b := av\lambda(b)$. 
Simple Two-sided Vector Spaces: Examples
Example 1

- $k = \mathbb{R}$, $L = \mathbb{C}$, $G = \text{Gal}(\overline{L}/L) = \{\text{id}\}$
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$p \geq 3$ prime, $\zeta = \text{a primative } p\text{th root of unity.}$
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- \( p \geq 3 \) prime, \( \zeta = \text{a primative } p\text{th root of unity} \).
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- $$p \geq 3$$ prime, $$\zeta$$ = a primitive $$p$$th root of unity.
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Right dual of $V$

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Adam Nyman
Duals

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If $\sigma \in \text{Gal}(L/k)$ then $^*L_\sigma \cong L_\sigma^* \cong L_{\sigma^{-1}}$
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**Example**

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**Theorem (Hart and N. 2012)**
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Suppose $V \cong V(\lambda)$, and let $\overline{\lambda} : \overline{L} \rightarrow \overline{L}$ be a lift of $\lambda$. Let $\mu := (\overline{\lambda})^{-1}|_L$. Then
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$*V \cong V^* \cong V(\mu)$. 

Adam Nyman
Theme Revisited

Adam Nyman
If $V$ is not simple,
If $V$ is not simple, study

$$\{\sigma, \tau\} \leadsto \mathbb{P}^{n.c.}(L_{\sigma} \oplus L_{\tau})$$
If $V$ is not simple, study

$$\{\sigma, \tau\} \mapsto P_{n.c.}^{\cdot}(L_{\sigma} \oplus L_{\tau})$$

If $V$ is simple,
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$$\{\sigma, \tau\} \sim \mathbb{P}^{n.c.}(L_\sigma \oplus L_\tau)$$

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$$\lambda \sim \mathbb{P}^{n.c.}(V(\lambda))$$
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Arithmetic $\leadsto$ Noncommutative geometry
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Arithmetic $\leadsto$ Noncommutative geometry

Questions
If $V$ is not simple, study
\[ \{\sigma, \tau\} \leadsto \mathbb{P}^{n.c.}(L_\sigma \oplus L_\tau) \]

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Arithmetic $\leadsto$ Noncommutative geometry

Questions

1. For which arithmetic data are associated spaces isomorphic?
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Arithmetic $\leadsto$ Noncommutative geometry

Questions

1. For which arithmetic data are associated spaces isomorphic?
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Theme Revisited

If $V$ is not simple, study

$$\{\sigma, \tau\} \leadsto \mathbb{P}^{n.c.}(L_\sigma \oplus L_\tau)$$

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Arithmetic $\leadsto$ Noncommutative geometry

Questions

1. For which arithmetic data are associated spaces isomorphic?
2. If they are isomorphic, what are the isomorphisms?
3. What is the relationship between the arithmetic data and the automorphism groups?
Part 3

Noncommutative Symmetric Algebras
Suppose

- $V$ has rank two.
- $\{x, y\}$ is *simultaneous* basis for $V$. 
Suppose

- $V$ has rank two.
- $\{x, y\}$ is simultaneous basis for $V$.

Construct n.c. ring $S^{n.c.}(V)$ which specializes to

$$S(V) := \frac{L \oplus V \oplus V \otimes 2 \oplus \cdots}{(x \otimes y - y \otimes x)}$$

when $V$ is $L$-central.
Suppose
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Construct n.c. ring $\mathcal{S}^{n.c.}(V)$ which specializes to

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when $V$ is $L$-central.

Should have expected left and right Hilbert series
Define

\[ S^{n.c.}(V) := \frac{L \oplus V \oplus V^{\otimes 2} \oplus \cdots}{(x \otimes y - y \otimes x)} \]
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Problem
Too many relations.
There exists canonical $\eta_0 : L \rightarrow V \otimes_L V^*$:
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There exists canonical $\eta_0 : L \rightarrow V \otimes_L V^*$: If $\delta_x \in \text{Hom}_L(V_L, L)$ is dual to $x$ etc. then

$$\eta_0(a) := a(x \otimes \delta_x + y \otimes \delta_y).$$
There exists canonical $\eta_0 : L \to V \otimes_L V^*$: If $\delta_x \in \text{Hom}_L(V_L, L)$ is dual to $x$ etc. then

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$\eta_0$ independent of choices.
There exists canonical $\eta_0 : L \to V \otimes_L V^*$: If $\delta_x \in \text{Hom}_L(V_L, L)$ is dual to $x$ etc. then

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$\eta_0$ independent of choices. Define

$$S^{n.c.}(V) := L \oplus V \oplus \frac{V \otimes_L V^*}{\text{im} \eta_0} \oplus \frac{V \otimes V^* \otimes V^{**}}{\text{im} \eta_0 \otimes V^{**} + V \otimes \text{im} \eta_1} \oplus \cdots$$
There exists canonical $\eta_0 : L \to V \otimes_L V^*$: If $\delta_x \in \text{Hom}_L(V_L, L)$ is dual to $x$ etc. then

$$\eta_0(a) := a(x \otimes \delta_x + y \otimes \delta_y).$$

$\eta_0$ independent of choices. Define

$$S^{n.c.}(V) := L \oplus V \oplus \frac{V \otimes_L V^*}{\text{im} \eta_0} \oplus \frac{V \otimes V^* \otimes V^{**}}{\text{im} \eta_0 \otimes V^{**} + V \otimes \text{im} \eta_1} \oplus \cdots$$

**Problem**

No natural multiplication: if $x, y \in V$, $x \cdot y$ not in $\frac{V \otimes V^*}{\text{im} \eta_0}$. 

Adam Nyman
Z-algebras (Bondal and Polishchuk (1993))
A ring $A$ is a $\mathbb{Z}$-algebra if
$\mathbb{Z}$-algebras (Bondal and Polishchuk (1993))

A ring $A$ is a $\mathbb{Z}$-algebra if

- $\exists$ vector space decomp $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$,
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Remark: $A$ does not have a unity and is not a domain.
A ring $A$ is a $\mathbb{Z}$-algebra if

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Remark: $A$ does not have a unity and is not a domain.

Example

If $(\mathcal{O}(n))_{n \in \mathbb{Z}}$ is seq. of objects in a category $A$, then

$$A_{ij} = \text{Hom}_A(\mathcal{O}(-j), \mathcal{O}(-i))$$

with mult. = composition makes $\oplus_{i,j \in \mathbb{Z}} A_{ij}$ a $\mathbb{Z}$-algebra
Attempt 3: $S^{n.c.}(V)$ is a $\mathbb{Z}$-algebra
Definition of $S^{n.c.}(V)$ (Van den Bergh (2000))

\[ S^{n.c.}(V)_{ij} = \frac{V^i \otimes_L \cdots \otimes_L V^{j-1} \ast}{\text{relns. gen. by } \eta_i} \text{ for } j > i, \]
Attempt 3: $S^{n.c.}(V)$ is a $\mathbb{Z}$-algebra

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Adam Nyman
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Adam Nyman
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Adam Nyman
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More generally, if
- $X$ is a smooth scheme of finite type over a $k$
Definition of $\mathbb{S}^{n.c.}(V)$ (Van den Bergh (2000))

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- $\mathcal{E}$ is a locally free rank $n$ $\mathcal{O}_X$-bimodule
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Definition of $S^{n.c.}(V)$ (Van den Bergh (2000))

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More generally, if

- $X$ is a smooth scheme of finite type over a $k$
- $E$ is a locally free rank $n$ $\mathcal{O}_X$-bimodule

Van den Bergh defines $S^{n.c.}(E)$.
Relation to $S(V)$
If $V$ is $L$-central, $S^{n.c.}(V) \neq S(V)$. 

Adam Nyman
Relation to $\mathcal{S}(V)$

If $V$ is $L$-central, $\mathcal{S}^{n.c.}(V) \neq \mathcal{S}(V)$.

If $A$ is a $\mathbb{Z}$-algebra,
If $V$ is $L$-central, $S_{\text{n.c.}}(V) \neq S(V)$.

If $A$ is a $\mathbb{Z}$-algebra,

- if $i \in \mathbb{Z}$ let $A(i)_{jk} := A_{j+i,k+i}$. 
If $V$ is $L$-central, $S^{n.c.}(V) \neq S(V)$.

If $A$ is a $\mathbb{Z}$-algebra,

- if $i \in \mathbb{Z}$ let $A(i)_{jk} := A_{j+i,k+i}$.
- $A$ is $i$-periodic if $A \cong A(i)$. 

Adam Nyman
If $V$ is $L$-central, $S^{n.c.}(V) \neq S(V)$.

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If $B$ is $\mathbb{Z}$-graded algebra, define $\check{B}_{ij} := B_{j-i}$. 
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If $B$ is $\mathbb{Z}$-graded algebra, define $\tilde{B}_{ij} := B_{j-i}$.

**Theorem (Van den Bergh (2000))**

If $A$ is 1-periodic, then there exists a $\mathbb{Z}$-graded ring $\tilde{B}$ such that $A \cong \tilde{B}$.
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**Theorem (Van den Bergh (2000))**
If $A$ is 1-periodic, then there exists a $\mathbb{Z}$-graded ring $B$ such that $A \cong \tilde{B}$, and $\text{Gr}A \equiv \text{Gr}B$. 
If $V$ is $L$-central, $S^{n.c.}(V) \neq S(V)$.

If $A$ is a $\mathbb{Z}$-algebra,
- if $i \in \mathbb{Z}$ let $A(i)_{jk} := A_{j+i,k+i}$.
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If $B$ is $\mathbb{Z}$-graded algebra, define $\tilde{B}_{ij} := B_{j-i}$.

**Theorem (Van den Bergh (2000))**

If $A$ is 1-periodic, then there exists a $\mathbb{Z}$-graded ring $B$ such that $A \cong \tilde{B}$, and $\text{Gr}A \equiv \text{Gr}B$. It follows that if $V$ is $L$-central, then

$$\text{Gr}S^{n.c.}(V) \equiv \text{Gr}S(V).$$
Part 4

Arithmetic Noncommutative Projective Lines
Basic Properties

- $V$ a rank 2 ($k$-central) two-sided vector space $/L$
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- $\text{Tors}^{n.c.}(V)$ = full subcat. of $\text{Gr}^{n.c.}(V)$ of direct limits of right bounded modules
Basic Properties

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- $\text{Tors}^{n.c.}(V) = \text{full subcat. of Gr}^{n.c.}(V)$ of direct limits of right bounded modules
- $\mathbb{P}^{n.c.}(V) := \text{Gr}^{n.c.}(V) / \text{Tors}^{n.c.}(V)$,
Basic Properties

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Theorem

The noncommutative space $\mathbb{P}^{n.c.}(V)$
V a rank 2 \((k\text{-central})\) two-sided vector space \(/L\)

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**Theorem**

The noncommutative space \(\mathbb{P}^{n.c.}(V)\)

- is a locally noetherian category (Van den Bergh (2000)),

Adam Nyman
Basic Properties

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Theorem

The noncommutative space $\mathbb{P}^{n.c.}(V)$
- is a locally noetherian category (Van den Bergh (2000)),
- is Ext-finite (N. (2004)).
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Theorem

The noncommutative space $\mathbb{P}^{n.c.}(V)$
- is a locally noetherian category (Van den Bergh (2000)),
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Basic Properties

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- has homological dimension 1 (Chan and N. (2009)), and
- has a tilting object $\mathcal{T}$. 

Adam Nyman
Motivation: Birational Classification of Noncommutative Surfaces
Conjecture (Artin)

Every noncommutative surface not finite over its center is birational to a noncommutative ruled surface.
Conjecture (Artin)

Every noncommutative surface not finite over its center is birational to a noncommutative ruled surface.

If \( C, C' \) are \( \mathbb{Z} \)-graded,

\[
\text{Proj} \, C \text{ birational to } \text{Proj} \, C'
\]

\textbf{means} deg. 0 comp. of skew field of \( C \) equals that of \( C' \).
Motivation: Birational Classification of Noncommutative Surfaces

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Relationship to $\mathbb{P}^{n.c.}(V)$

Generic fibre of noncommutative ruled surface $\cong \mathbb{P}^{n.c.}(V)$ where $V$ is two-sided over $L$
Motivation: Birational Classification of Noncommutative Surfaces

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Relationship to $\mathbb{P}^{n.c.}(V)$

Generic fibre of noncommutative ruled surface $\cong \mathbb{P}^{n.c.}(V)$ where $V$ is two-sided over $L = \text{function field of smooth curve}$
Motivation: Birational Classification of Noncommutative Surfaces

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Relationship to $\mathbb{P}^{n.c.}(V)$

Generic fibre of noncommutative ruled surface $\cong \mathbb{P}^{n.c.}(V)$ where $V$ is two-sided over $L =$ function field of smooth curve

Birational invariants of noncommutative projective lines $\mathbb{P}^{n.c.}(V)$ may suggest birational invariants of a noncommutative surface.
“The motivation for a physicist to study 1-dimensional problems is best illustrated by the story of the man who, returning home late at night after an alcoholic evening, was scanning the ground for his key under a lamppost; he knew, to be sure, that he had dropped it somewhere else, but only under the lamppost was there enough light to conduct a proper search.” –F. Calogero
“The motivation for a physicist to study 1-dimensional problems is best illustrated by the story of the man who, returning home late at night after an alcoholic evening, was scanning the ground for his key under a lamppost; he knew, to be sure, that he had dropped it somewhere else, but only under the lamppost was there enough light to conduct a proper search.” – F. Calogero

Thanks Thomas Nevins.
$\mathbb{P}^{n.c.}(V)$ is Integral
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Let $X$ = locally noetherian noncommutative space.
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**Definition (S.P. Smith (2001))**
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$X$ is **integral** if $\exists$ indecomposable injective $\mathcal{E}_X$ (a **big injective**) such that
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$X$ is **integral** if $\exists$ indecomposable injective $\mathcal{E}_X$ (a **big injective**) such that

1. $\operatorname{End}_X(\mathcal{E}_X)$ is a division ring and
Let $X =$ locally noetherian noncommutative space.

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$X$ is **integral** if $\exists$ indecomposable injective $E_X$ (a **big injective**) such that

1. $\text{End}_X(E_X)$ is a division ring and
2. every object of $X$ is a subquotient of $\bigoplus E_X$. 
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A noetherian scheme $Y$ is integral in the above sense iff $Y$ is integral in the usual sense, and $E_{\mathcal{Qcoh}Y}$ is the constant sheaf with sections $= k(Y)$. 
$\mathbb{P}^{n.c.}(V)$ is Integral

Let $X = \text{locally noetherian noncommutative space}.$

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**Theorem (N. 2013)**

The noncommutative space $\mathbb{P}^{n.c.}(V)$ is integral.
$M \in X$ is **torsion** if $\text{Hom}_X (M, \mathcal{E}_X) = 0$. 
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rank $M :=$ length of $\text{Hom}_X(M, \mathcal{E}_X)$ as left $\text{End}_X(\mathcal{E}_X)$-module.
Classification of Vector Bundles

- $M \in X$ is **torsion** if $\text{Hom}_X(M, \mathcal{E}_X) = 0$.
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**Definition**

Vector bundles $/ X =$
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Classification of Vector Bundles

- $M \in X$ is torsion if $\text{Hom}_X(M, E_X) = 0$.
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**Definition**

Vector bundles $/X = $ finite rank torsion-free modules.

- Let $e_i S^{n,c}(V) := \bigoplus_{j \in \mathbb{Z}} S^{n,c}(V)_{ij} \in \text{Gr} S^{n,c}(V)$. 

Adam Nyman
Classification of Vector Bundles

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**Definition**

Vector bundles \( /X = \) finite rank torsion-free modules.

- Let \( e_i^{S_n.\cdot}(V) := \bigoplus_{j \in \mathbb{Z}} S_j^{n.\cdot}(V)_{ij} \in \text{Gr} S^{n.\cdot}(V) \).
- Let \( \pi : \text{Gr} S^{n.\cdot}(V) \to P^{n.\cdot}(V) \) be the quotient functor.
Classification of Vector Bundles

- $M \in X$ is **torsion** if $\text{Hom}_X(M, \mathcal{E}_X) = 0$.
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- Let $\mathcal{O}(i) :=$
\[ M \in X \text{ is torsion if } \text{Hom}_X(M, \mathcal{E}_X) = 0. \]

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- Let \( \mathcal{O}(i) := \pi(e_{-i} S^{n.c.}(V)). \)
Classification of Vector Bundles

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**Theorem (N. 2013)**

Every vector bundle over $\mathbb{P}^{n.c.}(V)$ is a direct sum of line bundles.
Classification of Vector Bundles

- $M \in X$ is **torsion** if $\text{Hom}_X(M, \mathcal{E}_X) = 0$.
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**Definition**

Vector bundles $/X = \text{finite rank torsion-free modules}$.

- Let $e_i \mathbb{S}^{n.c.}(V) := \bigoplus_{j \in \mathbb{Z}} \mathbb{S}^{n.c.}(V)_{ij} \in \text{Gr} \mathbb{S}^{n.c.}(V)$.
- Let $\pi : \text{Gr} \mathbb{S}^{n.c.}(V) \to \mathbb{P}^{n.c.}(V)$ be the quotient functor.
- Let $O(i) := \pi(e_{-i} \mathbb{S}^{n.c.}(V))$.

**Theorem (N. 2013)**

Every vector bundle over $\mathbb{P}^{n.c.}(V)$ is a direct sum of line bundles. The line bundles are $\{O(i)\}_{i \in \mathbb{Z}}$. 

Adam Nyman
Part 5

Classification of Noncommutative Projective Lines
Classification Theorem Version 1

Theorem (N. (2013))

\[ \mathbb{P}^{n.c.} (V) \equiv_k \mathbb{P}^{n.c.} (W) \] if and only if
Theorem (N. (2013))

\[ \mathbb{P}^{n.c.}(V) \equiv_k \mathbb{P}^{n.c.}(W) \text{ if and only if there exists } \sigma, \tau \in \text{Gal}(L/k) \text{ such that either} \]

\[ V \cong L_\sigma \otimes_L W \otimes_L L_\tau \]
Theorem (N. (2013))

\[ \mathbb{P}^{n.c.}(V) \equiv_k \mathbb{P}^{n.c.}(W) \text{ if and only if there exists } \sigma, \tau \in \text{Gal}(L/k) \text{ such that either} \]

\[ V \cong L_\sigma \otimes_L W \otimes_L L_\tau \text{ or } V \cong L_\sigma \otimes_L W^* \otimes_L L_\tau. \]
Theorem (N. (2013))

\[ \mathbb{P}^{n.c.}(V) \cong_k \mathbb{P}^{n.c.}(W) \] if and only if there exists \( \sigma, \tau \in \text{Gal}(L/k) \) such that either

\[ V \cong L_\sigma \otimes_L W \otimes_L L_\tau \] or \[ V \cong L_\sigma \otimes_L W^* \otimes_L L_\tau. \]

(\( \Leftarrow \)) proven in greater generality by I. Mori.
Theorem (N. (2013))

Suppose char $k \neq 2$. Then $\mathbb{P}^{n.c.}(V_1) \equiv \mathbb{P}^{n.c.}(V_2)$ if and only if
Theorem (N. (2013))

Suppose char \( k \neq 2 \). Then \( \mathbb{P}^{n.c.}(V_1) \equiv \mathbb{P}^{n.c.}(V_2) \) if and only if

Case 1: \( \exists \sigma_i \in \text{Gal}(L/k) \) such that

\[
V_i \cong L_{\sigma_i} \oplus L_{\sigma_i}.
\]
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Adam Nyman
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**Case 2:** \( \exists \sigma_i, \tau_i \in \text{Gal}(L/k) \), with \( \sigma_i \neq \tau_i \),

\[
V_i \cong L_{\sigma_i} \oplus L_{\tau_i}.
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**Case 2:** $\exists \sigma_i, \tau_i \in \text{Gal}(L/k)$, with $\sigma_i \neq \tau_i$,

$$V_i \cong L_{\sigma_i} \oplus L_{\tau_i}$$

and under action of $\text{Gal}(L/k)^2$ on itself defined by

$$(\alpha, \beta) \cdot (\sigma, \tau) := (\alpha \sigma \beta^{-1}, \alpha \tau \beta^{-1})$$
Suppose $\text{char } k \neq 2$. Then $\mathbb{P}^{n.c.}(V_1) \equiv \mathbb{P}^{n.c.}(V_2)$ if and only if

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and under action of $\text{Gal}(L/k)^2$ on itself defined by

$$(\alpha, \beta) \cdot (\sigma, \tau) := (\alpha \sigma \beta^{-1}, \alpha \tau \beta^{-1})$$

$\mathcal{O}_{(\sigma_1, \tau_1)} \cap \{(\sigma_2, \tau_2), (\sigma_2^{-1}, \tau_2^{-1}), (\tau_2, \sigma_2), (\tau_2^{-1}, \sigma_2^{-1})\} \neq \emptyset.$
Theorem (cont.)

Let $G := \text{Gal}(\overline{L}/L)$. Suppose $\text{char } k \neq 2$. Then

$\mathbb{P}^{n.c.}(V_1) \equiv \mathbb{P}^{n.c.}(V_2)$ if and only if

**Case 3:** $\exists \lambda_i \in \text{Emb}(L)$ of $G$-orbit size two, such that

$$V_i \cong V(\lambda_i),$$
Theorem (cont.)

Let $G := \text{Gal}(\overline{L}/L)$. Suppose $\text{char } k \neq 2$. Then $P^{n.c.}(V_1) \equiv P^{n.c.}(V_2)$ if and only if

**Case 3:** $\exists \lambda_i \in \text{Emb}(L)$ of $G$-orbit size two, such that

$$V_i \cong V(\lambda_i),$$

and under the action of $\text{Gal}(L/k)^2$ on $\text{Emb}(L)$ defined by

$$(\alpha, \beta) \cdot \lambda := \alpha \lambda \beta^{-1},$$

Either
Classification Theorem Version 2, Case 3

Theorem (cont.)

Let \( G := \text{Gal}(\overline{L}/L) \). Suppose \( \text{char } k \neq 2 \). Then
\[
\mathbb{P}^{n.c.}(V_1) \equiv \mathbb{P}^{n.c.}(V_2)
\]
if and only if

**Case 3:** \( \exists \, \lambda_i \in \text{Emb}(L) \) of \( G \)-orbit size two, such that

\[
V_i \cong V(\lambda_i),
\]

and under the action of \( \text{Gal}(L/k)^2 \) on \( \text{Emb}(L) \) defined by

\[
(\alpha, \beta) \cdot \lambda := \alpha \lambda \beta^{-1},
\]

Either

- \( O_{\lambda_1} \cap \lambda_2^G \neq \emptyset \) or
Theorem (cont.)

Let \( G := \text{Gal}(\overline{L}/L) \). Suppose \( \text{char } k \neq 2 \). Then
\[
\mathbb{P}^{n.c.}(V_1) \equiv \mathbb{P}^{n.c.}(V_2)
\]
if and only if

**Case 3:** \( \exists \lambda_i \in \text{Emb}(L) \) of \( G \)-orbit size two, such that
\[
V_i \cong V(\lambda_i),
\]
and under the action of \( \text{Gal}(L/k)^2 \) on \( \text{Emb}(L) \) defined by
\[
(\alpha, \beta) \cdot \lambda := \alpha \lambda \beta^{-1},
\]
Either
- \( \mathcal{O}_{\lambda_1} \cap \lambda_2^G \neq \emptyset \) or
- \( \mathcal{O}_{\lambda_1} \cap \mu_2^G \neq \emptyset \) where \( \mu_2 = (\lambda_2)^{-1}|_L \).
Part 6

Classification of Isomorphisms $\mathbb{P}^{n.c.}(V) \rightarrow \mathbb{P}^{n.c.}(W)$
$\phi : V \xrightarrow{\sim} W$ induces $\phi : S^{n.c.}(V) \xrightarrow{\sim} S^{n.c.}(W)$. 
\( \phi : V \xrightarrow{\sim} W \) induces \( \phi : S^{n.c.}(V) \xrightarrow{\sim} S^{n.c.}(W) \).

**The equivalence \( \Phi \)**
\[ \phi : V \xrightarrow{\sim} W \text{ induces } \phi : \mathbb{S}^{n.c.}(V) \xrightarrow{\sim} \mathbb{S}^{n.c.}(W). \]

The equivalence \( \Phi \)

Definition of \( \Phi : \text{Gr}\mathbb{S}^{n.c.}(V) \rightarrow \text{Gr}\mathbb{S}^{n.c.}(W) : \)
Canonical Equivalences 1

$\phi : V \overset{\sim}{\rightarrow} W$ induces $\phi : S^{n.c.}(V) \overset{\sim}{\rightarrow} S^{n.c.}(W)$.

The equivalence $\Phi$

Definition of $\Phi : GrS^{n.c.}(V) \rightarrow GrS^{n.c.}(W)$:

- $\Phi(M)_i := M_i$ as a set, with $S^{n.c.}(W)$-module structure

$$
\Phi(M)_i \otimes S^{n.c.}(W)_{ij} \overset{1 \otimes \phi^{-1}}{\rightarrow} \Phi(M)_i \otimes S^{n.c.}(V)_{ij} \overset{\mu}{\rightarrow} \Phi(M)_j.
$$
φ : V \rightarrow W induces \phi : S^{n.c.}(V) \rightarrow S^{n.c.}(W).

The equivalence \Phi

Definition of \Phi : GrS^{n.c.}(V) \rightarrow GrS^{n.c.}(W):

- \Phi(M)_i := M_i as a set, with S^{n.c.}(W)-module structure

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\]

- If \( f : M \rightarrow N \) we define \( \Phi(f)_i(m) = f(m) \).
$\phi : V \xrightarrow{\sim} W$ induces $\phi : S^{n.c.}(V) \xrightarrow{\sim} S^{n.c.}(W)$.

**The equivalence $\Phi$**

**Definition of $\Phi : \text{Gr}S^{n.c.}(V) \rightarrow \text{Gr}S^{n.c.}(W)$:**

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\Phi(M)_i \otimes S^{n.c.}(W)_{ij} \xrightarrow{1 \otimes \phi^{-1}} \Phi(M)_i \otimes S^{n.c.}(V)_{ij} \xrightarrow{\mu} \Phi(M)_j.
\]

- If $f : M \rightarrow N$ we define $\Phi(f)_i(m) = f(m)$.

$\Phi$ descends uniquely to an equivalence $\Phi : \mathbb{P}^{n.c.}(V) \rightarrow \mathbb{P}^{n.c.}(W)$. 

Adam Nyman
For $i \in \mathbb{Z}$, let $\sigma_i \in \text{Gal}(L/k)$,
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Let $\sigma := \{\sigma_i\}_{i \in \mathbb{Z}}$, and
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If $A$ denotes a $\mathbb{Z}$-algebra, let $A_\sigma$ denote the $\mathbb{Z}$-algebra with

$$A_{\sigma,ij} := L_{\sigma_i^{-1}} \otimes A_{ij} \otimes L_{\sigma_j}$$

and with multiplication induced by that of $A$. 
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The equivalence $T_\sigma$ (Van den Bergh)

Definition of $T_\sigma : \text{Gr}A \rightarrow \text{Gr}A_\sigma$:
For $i \in \mathbb{Z}$, let $\sigma_i \in \text{Gal}(L/k)$,

Let $\sigma := \{\sigma_i\}_{i \in \mathbb{Z}}$, and

If $A$ denotes a $\mathbb{Z}$-algebra, let $A_\sigma$ denote the $\mathbb{Z}$-algebra with

$$A_{\sigma,ij} := L_{\sigma_i}^{-1} \otimes A_{ij} \otimes L_{\sigma_j}$$

and with multiplication induced by that of $A$.

The equivalence $T_\sigma$ (Van den Bergh)

Definition of $T_\sigma : \text{Gr} A \rightarrow \text{Gr} A_\sigma$:

$T_\sigma(M)_i := M_i \otimes L_{\sigma_i}$ with multiplication induced by that of $A$, and
For $i \in \mathbb{Z}$, let $\sigma_i \in \text{Gal}(L/k)$,

Let $\sigma := \{\sigma_i\}_{i \in \mathbb{Z}}$, and

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**Definition of $T_\sigma : \text{Gr}A \rightarrow \text{Gr}A_\sigma$:**

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The equivalence $T_\sigma$ (Van den Bergh)

Definition of $T_\sigma : \text{Gr}A \to \text{Gr}A_\sigma$:

- $T_\sigma(M)_i := M_i \otimes L_{\sigma_i}$ with multiplication induced by that of $A$, and
- If $f : M \to N$ we define $T_\sigma(f)_i = f_i \otimes L_{\sigma_i}$.

$T_\sigma$ descends uniquely to an equivalence $T_\sigma : \text{Proj}A \to \text{Proj}A_\sigma$. 

Adam Nyman
A Special Twist

Adam Nyman
A Special Twist

For $\delta, \tau \in \text{Gal}(L/k)$

$$\zeta_i = \begin{cases} 
\delta & \text{if } i \text{ is even} \\
\tau & \text{if } i \text{ is odd},
\end{cases}$$
For $\delta, \tau \in \text{Gal}(L/k)$

$$\zeta_i = \begin{cases} 
\delta & \text{if } i \text{ is even} \\
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In this case there is a canonical isomorphism

$$\mathcal{S}^{n.c.}(V)_{\zeta} \rightarrow \mathcal{S}^{n.c.}(L_{\delta^{-1}} \otimes V \otimes L_{\tau}).$$
A Special Twist

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$$\mathcal{S}^{n.c.}(V)_{\zeta} \rightarrow \mathcal{S}^{n.c.}(L_{\delta^{-1}} \otimes V \otimes L_{\tau}).$$

Notation

$$T_{\delta, \tau} : \mathbb{P}^{n.c.}(V) \rightarrow \mathbb{P}^{n.c.}(L_{\delta^{-1}} \otimes V \otimes L_{\tau})$$
Shift Functor

Definition of $[i] : \text{Gr}^{n.c.}(V) \to \text{Gr}^{n.c.}(V) \ (i \in \mathbb{Z})$:

- $M[i]_j := M_{j+i}$ with multiplication induced from mult. on $M$
- If $f : M \to N$, $f[i]_j = f_{j+i}$. 

Adam Nyman
Shift Functor

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Problem

If \(i\) is odd, \(M[i]\) does not inherit \(\mathbb{S}^{n.c.}(V)\)-module mult. from \(M\)!
Shift Functor

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Problem

If $i$ is odd, $M[i]$ does not inherit $S^{n.c.}(V)$-module mult. from $M$!
But $M[i]$ does have a $S^{n.c.}(V^*)$-module structure (I. Mori)
Shift Functor

Definition of \([i] : \text{Gr}\mathbb{S}^{n.c.}(V) \rightarrow \text{Gr}\mathbb{S}^{n.c.}(V) \ (i \in \mathbb{Z})\):

- \(M[i]_j \coloneqq M_{j+i}\) with multiplication induced from mult. on \(M\)
- If \(f : M \rightarrow N\), \(f[i]_j = f_{j+i}\).

Problem

If \(i\) is odd, \(M[i]\) does not inherit \(\mathbb{S}^{n.c.}(V)\)-module mult. from \(M\)!
But \(M[i]\) does have a \(\mathbb{S}^{n.c.}(V^*)\)-module structure (I. Mori)

\[
[i] : \mathbb{P}^{n.c.}(V) \rightarrow \begin{cases} 
\mathbb{P}^{n.c.}(V) & \text{if } i \text{ is even} \\
\mathbb{P}^{n.c.}(V^*) & \text{if } i \text{ is odd}
\end{cases}
\]
Classification of Isomorphisms
Classification of Isomorphisms

Theorem (N. (2013))

If $F : \mathbb{P}^{n,c}(V) \to \mathbb{P}^{n,c}(W)$ is $k$-linear equivalence, there exists
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Classification of Isomorphisms

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Theorem (N. (2013))

If $F : \mathbb{P}^n.c.(V) \to \mathbb{P}^n.c.(W)$ is $k$-linear equivalence, there exists
- $i \in \mathbb{Z}$,
- $\sigma, \tau \in \text{Gal}(L/k)$, and
- an isomorphism $\phi : L_{\sigma^{-1}} \otimes_L V \otimes L L_{\tau} \to W^{-i^*}$
Theorem (N. (2013))

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such that

$$F \cong [i] \circ \Phi \circ T_{\sigma, \tau}.$$
Theorem (N. (2013))

If \( F : \mathbb{P}^{n,c}(V) \to \mathbb{P}^{n,c}(W) \) is \( k \)-linear equivalence, there exists

- \( i \in \mathbb{Z} \),
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such that

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F \cong [i] \circ \Phi \circ T_{\sigma,\tau}.
\]

Furthermore,

- \( i, \sigma \) and \( \tau \) are unique up to natural equivalence and
Theorem (N. (2013))

If $F : \mathbb{P}^{n.c.}(V) \to \mathbb{P}^{n.c.}(W)$ is $k$-linear equivalence, there exists

- $i \in \mathbb{Z}$,
- $\sigma, \tau \in \text{Gal}(L/k)$, and
- an isomorphism $\phi : L_{\sigma^{-1}} \otimes_L V \otimes_L L_\tau \to W^{-i^*}$

such that

$$F \cong [i] \circ \Phi \circ T_{\sigma, \tau}.$$

Furthermore,

- $i$, $\sigma$ and $\tau$ are unique up to natural equivalence and
- $\Phi \equiv \Phi' \iff$ there exist $\alpha, \beta \in L^*$ such that
  $$\phi' \circ \phi^{-1}(w) = \alpha \cdot w \cdot \beta \text{ for all } w \in W^{-i^*}$$
Classification of Isomorphisms

**Theorem (N. (2013))**

If $F : \mathbb{P}^{n,c}(V) \to \mathbb{P}^{n,c}(W)$ is $k$-linear equivalence, there exists

- $i \in \mathbb{Z}$,
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Part 7

Automorphism Groups
Aut $\mathbb{P}^{n.c.}(V)$, Stab $V$ and Aut $V$
The group $\text{Aut } \mathbb{P}^{n.c.}(V)$

$\text{Aut } \mathbb{P}^{n.c.}(V) :=$ the set equivalence classes of $k$-linear shift-free equivalences $\mathbb{P}^{n.c.}(V) \rightarrow \mathbb{P}^{n.c.}(V)$, with composition induced by composition of functors.
The group $\text{Aut } \mathbb{P}^{n.c.}(V)$

$\text{Aut } \mathbb{P}^{n.c.}(V) :=$ the set equivalence classes of $k$-linear shift-free equivalences $\mathbb{P}^{n.c.}(V) \to \mathbb{P}^{n.c.}(V)$, with composition induced by composition of functors.

To describe it: need

**Definition of Stab $V$**

$\text{Stab } V =$ subgroup of $\text{Gal } (L/k) \times \text{Gal } (L/k)$ consisting of $(\sigma, \tau)$ such that $L_{\sigma^{-1}} \otimes_L V \otimes_L L_{\tau} \cong V$
The group $\text{Aut } \mathbb{P}^{n.c.}(V)$

$\text{Aut } \mathbb{P}^{n.c.}(V) :=$ the set equivalence classes of $k$-linear shift-free equivalences $\mathbb{P}^{n.c.}(V) \rightarrow \mathbb{P}^{n.c.}(V)$, with composition induced by composition of functors.

To describe it: need

**Definition of $\text{Stab } V$**

$\text{Stab } V = \text{subgroup of } \text{Gal } (L/k) \times \text{Gal } (L/k) \text{ consisting of } (\sigma, \tau) \text{ such that } L_{\sigma^{-1}} \otimes_L V \otimes_L L_{\tau} \cong V$

**Definition of $\text{Aut } V$**

$\text{Aut } V = \text{the set of isomorphisms } V \rightarrow V$
The group $\text{Aut } \mathbb{P}^{n.c.}(V)$

$\text{Aut } \mathbb{P}^{n.c.}(V) :=$ the set equivalence classes of $k$-linear shift-free equivalences $\mathbb{P}^{n.c.}(V) \rightarrow \mathbb{P}^{n.c.}(V)$, with composition induced by composition of functors.

To describe it: need

**Definition of Stab $V$**

$\text{Stab } V =$ subgroup of $\text{Gal } (L/k) \times \text{Gal } (L/k)$ consisting of $(\sigma, \tau)$ such that $L_{\sigma^{-1}} \otimes_L V \otimes_L L_{\tau} \cong V$

**Definition of Aut $V$**

$\text{Aut } V =$ the set of isomorphisms $V \rightarrow V$ modulo the relation defined by setting $\phi' \equiv \phi \Leftrightarrow$ there exist $\alpha, \beta \in L^*$ such that $\phi' \circ \phi^{-1}(v) = \alpha \cdot v \cdot \beta$ for all $v \in V$.  

Adam Nyman
The Automorphism Group

**Theorem (N. (2013))**

There exists homomorphism $\psi : \text{Stab } V \rightarrow \text{End } (\text{Aut } (V))$ such that
The Automorphism Group

Theorem (N. (2013))

There exists homomorphism $\psi : \operatorname{Stab} V \rightarrow \operatorname{End} (\operatorname{Aut} (V))$ such that

$$\operatorname{Aut} \mathbb{P}^{n.c.}(V) \cong \operatorname{Aut} V \rtimes_\psi \operatorname{Stab} V^{op}.$$
Let $V = L_{\sigma} \oplus L_{\sigma}$. Then
Let $V = L_\sigma \oplus L_\sigma$. Then
- $\text{Stab } V \cong \text{Gal } (L/k)$ and
Let $V = L_\sigma \oplus L_\sigma$. Then

- $\text{Stab } V \cong \text{Gal } (L/k)$ and
- $\text{Aut } V \cong \text{PGL}_2(L)$. 
Let $V = L_\sigma \oplus L_\sigma$. Then

- $\text{Stab } V \cong \text{Gal } (L/k)$ and
- $\text{Aut } V \cong \text{PGL}_2(L)$.

Then $\psi : \text{Stab } V \rightarrow \text{End } (\text{Aut } (V))$ is the homomorphism
Let $V = L_\sigma \oplus L_\sigma$. Then

- $\text{Stab } V \cong \text{Gal } (L/k)$ and
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Then $\psi : \text{Stab } V \to \text{End } (\text{Aut } (V))$ is the homomorphism

$$\psi : \text{Gal } (L/k) \to \text{End } (\text{PGL}_2(L))$$
Let \( V = L_\sigma \oplus L_\sigma \). Then
- \( \text{Stab } V \cong \text{Gal } (L/k) \) and
- \( \text{Aut } V \cong \text{PGL}_2(L) \).

Then \( \psi : \text{Stab } V \to \text{End } (\text{Aut } (V)) \) is the homomorphism

\[
\psi : \text{Gal } (L/k) \to \text{End } (\text{PGL}_2(L))
\]

defined by

\[
\psi(\sigma)[(a_{ij})] = [(\sigma(a_{ij}))]
\]
Let $V = L_\sigma \oplus L_\tau$ with $\sigma \neq \tau$. 
Let $V = L_\sigma \oplus L_\tau$ with $\sigma \neq \tau$. Then

- $\text{Stab } V = \{(g, h) | \{g^{-1}\sigma h, g^{-1}\tau h\} = \{\sigma, \tau\}\}$ and
Let $V = L_\sigma \oplus L_\tau$ with $\sigma \neq \tau$. Then

- $\text{Stab } V = \{(g, h) | \{g^{-1}\sigma h, g^{-1}\tau h\} = \{\sigma, \tau\}\}$ and

There are two types of elements in $\text{Stab } V$. 
Let \( V = L_\sigma \oplus L_\tau \) with \( \sigma \neq \tau \). Then

\[
\text{Stab } V = \{(g, h)|\{g^{-1}\sigma h, g^{-1}\tau h\} = \{\sigma, \tau\}\} \quad \text{and}
\]

There are two types of elements in \( \text{Stab } V \).

\[
\text{Aut } V \cong L^* \times L^*/\{(\alpha\sigma(\beta), \alpha\tau(\beta))|\alpha, \beta \in L^*\}
\]
Let $V = L_{\sigma} \oplus L_{\tau}$ with $\sigma \neq \tau$. Then

$\text{Stab } V = \{(g, h) | \{g^{-1}\sigma h, g^{-1}\tau h\} = \{\sigma, \tau\}\}$ and

There are two types of elements in $\text{Stab } V$.

$\text{Aut } V \cong L^* \times L^* / \{(\alpha\sigma(\beta), \alpha\tau(\beta)) | \alpha, \beta \in L^*\}$

Then $\psi : \text{Stab } V \rightarrow \text{End (Aut (V))}$ is defined by

$$\psi((g, h))[(a, b)] = [(g(a), g(b))]$$

if $g^{-1}\sigma h = \sigma$.
Let $V = L_\sigma \oplus L_\tau$ with $\sigma \neq \tau$. Then

- $\text{Stab } V = \{(g, h) | \{g^{-1}\sigma h, g^{-1}\tau h\} = \{\sigma, \tau\}\}$ and

There are two types of elements in $\text{Stab } V$.

- $\text{Aut } V \cong L^* \times L^*/\{(\alpha\sigma(\beta), \alpha\tau(\beta)) | \alpha, \beta \in L^*\}$

Then $\psi : \text{Stab } V \to \text{End } (\text{Aut } (V))$ is defined by

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In the special case that $V$ is not simple and $\text{Gal}(L/k)$ is cyclic the result was obtained by Kussin.
Let $V = V(\lambda) = 1L \vee \lambda(L)_\lambda$. 
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$$\text{Stab } V = \{(g, h) \in \text{Gal } (L/k) \times \text{Gal } (L/k) | (g^{-1}\lambda h)^G = \lambda^G\}$$

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Let $V = V(\lambda) = 1L \vee \lambda(L)\lambda$. Then

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**Lemma**

For each $(g, h) \in \text{Stab } V$, $\exists!$ field automorphism $\psi_{g,h} : L \triangledown \lambda(L) \rightarrow L \triangledown \lambda(L)$
Let $V = V(\lambda) = 1L \lor \lambda(L)^L$. Then

- Stab $V = \{(g, h) \in \text{Gal}(L/k) \times \text{Gal}(L/k) | (g^{-1} \lambda h)^G = \lambda^G\}$
  and
- $\text{Aut } V = (L \lor \lambda(L))^*/L^* \lambda(L)^*$

**Lemma**

For each $(g, h) \in \text{Stab } V$, $\exists!$ field automorphism $\psi_{g,h} : L \lor \lambda(L) \rightarrow L \lor \lambda(L)$ such that if $a \in L$ then $\psi_{g,h}(a) = g(a)$, and $\psi_{g,h}(\lambda(a)) = \lambda(h(a))$. 
Let $V = V(\lambda) = \lambda^1 L \vee \lambda(L)_\lambda$. Then

- $\text{Stab } V = \{(g, h) \in \text{Gal } (L/k) \times \text{Gal } (L/k)|(g^{-1}\lambda h)^G = \lambda^G\}$

and

- $\text{Aut } V = (L \vee \lambda(L))^*/L^* \lambda(L)^*$

**Lemma**

For each $(g, h) \in \text{Stab } V$, $\exists!$ field automorphism $\psi_{g,h} : L \vee \lambda(L) \to L \vee \lambda(L)$ such that if $a \in L$ then $\psi_{g,h}(a) = g(a)$, and $\psi_{g,h}(\lambda(a)) = \lambda(h(a))$.

Then $\psi : \text{Stab } V \to \text{End } (\text{Aut } (V))$ is the homomorphism defined by

$$\psi((g, h))[x] = [\psi_{g,h}(x)].$$
$\mathbb{P}^{n.c.}(V)$ is finite over its center (Kussin). No explicit description of center is known. Compute the center of $\mathbb{P}^{n.c.}(V(\lambda))$ as a function of $\lambda$. 

1
1. $\mathbb{P}^{n.c.}(V)$ is finite over its center (Kussin). No explicit description of center is known. Compute the center of $\mathbb{P}^{n.c.}(V(\lambda))$ as a function of $\lambda$.

2. Classify the spaces $\mathbb{P}^{n.c.}(V)$ up to derived equivalence.
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**Conjecture**

$$D^b(\mathbb{P}^{n.c.}(V)) \cong D^b(\mathbb{P}^{n.c.}(W)) \Rightarrow \mathbb{P}^{n.c.}(V) \cong \mathbb{P}^{n.c.}(W)$$
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**Conjecture**

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D^b(\mathbb{P}^{n.c.}(V)) \equiv D^b(\mathbb{P}^{n.c.}(W)) \Rightarrow \mathbb{P}^{n.c.}(V) \equiv \mathbb{P}^{n.c.}(W)
\]

and derived equivalences are induced by translations and equivalences

\[
\mathbb{P}^{n.c.}(V) \rightarrow \mathbb{P}^{n.c.}(W).
\]
Thank you for your attention!