Maps to Noncommutative $\mathbb{P}^1$ (w/ Daniel Chan)

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Conventions

- always work over a field $k$
- always work with right modules
Part 1

Noncommutative Curves (a very brief overview)
Noncommutative Space := Grothendieck Category =
- (k-linear) abelian category with
- exact direct limits and
- a generator.

The following are non-commutative spaces:
- Mod $R$, $R$ a ring
- Qcoh $X$, $X$ a scheme
- Proj $A := \text{Gr}A/\text{Tors}A$ where $A$ is $\mathbb{Z}$-graded

If $A$ is noetherian, proj$A :=$ noetherian objects in Proj $A$, and Proj $A$ is determined by proj$A$. 

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The Artin-Stafford Theorem

Theorem (Artin-Stafford (1995))

Let $k = \overline{k}$. If $A$ is connected GK-dimension 2 generated in degree 1, then there exists a projective curve $Y$ such that $\text{proj} A \equiv \text{coh} Y$.

Slogan

*Every noncommutative curve is commutative.*

Reiten-Van den Berg (2002) work more generally, classifying abelian connected $k$-linear categories which

- are noetherian,
- Ext-finite,
- hereditary, and
- satisfy Serre duality.
Let $A$ be coherent connected $\mathbb{N}$-graded algebra and let

- $\text{coh}A = \text{cat. of (graded right) coherent modules}$
- $\text{tors}A = \text{full subcat. of right-bounded modules}$.

**Definition (Polishchuk (2005))**

$\text{cohproj}A := \text{coh}A/\text{tors}A$

**Remark**

If $A$ is noetherian, $\text{cohproj}A \equiv \text{proj}A$. 
Theorem (Zhang (1998))

If $A$ is connected, gen. in degree 1 and regular of dim 2 then

$$A \cong k \langle x_1, \ldots, x_n \rangle / \langle b \rangle$$

where $n \geq 2$, $b = \sum_{i=1}^{n} x_i \sigma(x_{n-i+1})$ and $\sigma \in \text{Aut } k \langle x_1, \ldots, x_n \rangle$.

Theorem (Piontkovski (2008))

$n > 2$ implies $A$ is non-noetherian and coherent. If $\mathbb{P}_n^1 := \text{cohproj} A$, then $\mathbb{P}_n^1$ depends only on $n$. Furthermore, $\mathbb{P}_2^1 \equiv \text{coh}\mathbb{P}^1$. 
Let $X$ be smooth elliptic curve $/k = \mathbb{C}$.

**Theorem (Polishchuk (2002))**

For each $\theta \in \mathbb{R}$, $\exists$ $t$-structure on $D^b(X)$ w/heart $C^\theta$ such that

- $D^b(C^\theta) \equiv D^b(X)$,
- $C^\theta \equiv \text{cohproj}B$ for a right coherent $\mathbb{Z}$-algebra $B$. 
Our Goal

Suppose

- $X$ is scheme
- $\mathcal{L}$ is line-bundle on $X$ generated by $n + 1$ global sections.

Given $(X, \mathcal{L})$, there exists a morphism $f : X \to \mathbb{P}^n$.

Construction of $f$

$f$ is induced by taking Proj of a canonical map

$$\mathbb{S}(\text{Hom}(\mathcal{O}_X, \mathcal{L})) \to \bigoplus_i \text{Hom}(\mathcal{O}_X, \mathcal{L}^\otimes i)$$

Goal of Talk

Generalize to construct maps from noncommutative elliptic curves to $\mathbb{P}^n_1$. 
Part 2

Algebras Constructed from a Sequence
The orbit algebra of a sequence

If $\mathcal{L} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ is seq. of objects in a category $C$, then

$$(B_{\mathcal{L}})_{ij} = \text{Hom}(\mathcal{L}_{-j}, \mathcal{L}_{-i})$$

with mult. $= \text{composition}$ makes $B_{\mathcal{L}} = \bigoplus_{i,j \in \mathbb{Z}} (B_{\mathcal{L}})_{ij}$ a $\mathbb{Z}$-algebra, i.e.:

- $B_{ij}B_{jk} \subset B_{ik}$,
- $B_{ij}B_{kl} = 0$ for $k \neq j$, and
- the subalgebra $B_{ii}$ contains a unit.
Let $\mathcal{L} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ be seq. of objects in a category $C$.

**The noncommutative symmetric algebra of a sequence**

We let $S^{nc}(\mathcal{L})$ be the $\mathbb{Z}$-algebra generated by

$$S^{nc}(\mathcal{L})_{i,i+1} := \text{Hom}(\mathcal{L}_{-(i+1)}, \mathcal{L}_{-i})$$

with relations equal to the kernel of composition

$$\text{Hom}(\mathcal{L}_{-(i+1)}, \mathcal{L}_{-i}) \otimes \text{Hom}(\mathcal{L}_{-(i+2)}, \mathcal{L}_{-(i+1)}) \to \text{Hom}(\mathcal{L}_{-(i+2)}, \mathcal{L}_{-i}).$$

**An observation**

By construction, there is a morphism of $\mathbb{Z}$-algebras

$$S^{nc}(\mathcal{L}) \to B_\mathcal{L}.$$
Theorem (N (2019), Chan-N (2019))
Let \( M \) be a bimodule over a pair of division rings such \( M \) and \( M^* \) are left- and right-fd, and \( M \cong M^{**} \). Then \( \exists \mathcal{L} \) such that
\[
S^{nc}(M) \cong S^{nc}(\mathcal{L}).
\]

Theorem (Bondal-Polishchuk (1993), Van den Bergh (2011))
For every three-dimensional regular quadratic elliptic \( \mathbb{Z} \)-algebra \( A \), there exists an elliptic curve and a sequence of line bundles \( \mathcal{L} \) over it, such that
\[
A \cong S^{nc}(\mathcal{L}).
\]

It follows that the coordinate rings of noncommutative \( \mathbb{P}^1 \)'s and noncommutative \( \mathbb{P}^2 \)'s are noncommutative symmetric algebras.
Part 3

Helices
Definition of a Helix (The Hom-finite Case)

Let
- $C$ be abelian, Hom-finite, $k$-linear category
- $\mathcal{L} := (\mathcal{L}_i)_{i \in \mathbb{Z}}$ a sequence in $C$.

**Definition**

$\mathcal{L}$ is a helix if, $\forall \ i \in \mathbb{Z}$,

1. $\text{End}(\mathcal{L}_i) =: D_i$ is division ring
2. The canonical map
   
   $$\mathcal{L}_i \rightarrow \ast \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) \otimes_{D_{i+1}} \mathcal{L}_{i+1}$$
   
   is monomorphism with coker $\cong \mathcal{L}_{i+2}$
3. the canonical map

   $$\ast \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1}) \rightarrow \text{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_{i+2})$$

which exists by (2), is an isomorphism.
Let $\mathcal{C} = \text{coh} \mathbb{P}^1$ and let $\mathcal{L}$ be defined by $\mathcal{L}_i := \mathcal{O}_{\mathbb{P}^1}(i)$. Then $\mathcal{L}$ is a helix.

The map from (2) is the first nonzero arrow of the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(i) \to \mathcal{O}_{\mathbb{P}^1}(i+1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}(i+2) \to 0.$$ 

**Proposition**

Suppose $X$ is a projective variety and $\mathcal{L}$ is a line bundle generated by two global sections. Then $\{\mathcal{L}^\otimes i\}_{i \in \mathbb{Z}}$ is a helix in $\text{coh}X$. 

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Proposition (Chan-N. (2019))

If \( \mathcal{L} \) is a helix, then there is an isomorphism

\[
\mathbb{S}^{nc}(\text{Hom}(\mathcal{L}_{-1}, \mathcal{L}_0)) \xrightarrow{\text{ir}} \mathbb{S}^{nc}(\mathcal{L}).
\]

Thus, there exists a homomorphism

\[
\mathbb{S}^{nc}(\text{Hom}(\mathcal{L}_{-1}, \mathcal{L}_0)) \rightarrow B_{\mathcal{L}}
\]

which is an isomorphism in degree one.

This is the noncommutative version of the map

\[
\mathbb{S}(\text{Hom}(\mathcal{O}_X, \mathcal{L})) \rightarrow \bigoplus \text{Hom}(\mathcal{O}_X, \mathcal{L}^\otimes i)
\]

inducing \( f : X \rightarrow \mathbb{P}(\text{Hom}(\mathcal{O}_X, \mathcal{L})) \).
Part 4

Main Results
Theorem (Chan-N. (2019))

Suppose $\mathcal{L}$ is a helix such that

- $\text{Hom}(\mathcal{L}_j, \mathcal{L}_i) = 0 \ \forall j > i$
  
  ($\iff B_{\mathcal{L}}$ is connected),

- there exists $n > 0$ such that $\text{Ext}^1(\mathcal{L}_i, \mathcal{L}_{i+l}) = 0$ for all $l \geq n$
  
  (Serre vanishing for $\mathcal{L}$).

Then $\text{Tors}B_{\mathcal{L}} \subset \text{Gr}B_{\mathcal{L}}$ is localizing and the morphism from the proposition induces a map of spaces

$$\text{Proj}B_{\mathcal{L}} \rightarrow \text{Proj}^{nc}(\text{Hom}(\mathcal{L}_{-1}, \mathcal{L}_0)).$$
Let $X$ be a smooth elliptic curve over $k = \mathbb{C}$.

**Theorem (Chan-N. (2019))**

Let $n \geq 2$ and let $\mathcal{O}_X(1)$ correspond to a point $p \in X$. Then

1. there exists a unique helix $\mathcal{L}$ on $\text{coh}X$ with $\mathcal{L}_0 = \mathcal{O}_X$ and $\mathcal{L}_1 = \mathcal{O}_X(n)$,
2. the ring $B_\mathcal{L}$ is coherent and non-noetherian for $n > 2$,
3. $\text{cohproj}^{\text{Snc}}(\text{Hom}(\mathcal{L}_{-1}, \mathcal{L}_0)) \equiv \mathbb{P}^1_n$,
4. The map from the main theorem

   \[
   \text{cohproj}B_\mathcal{L} \to \mathbb{P}^1_n
   \]

   is a double cover, and
5. If $n = 2$ the map above is the adjoint pair of functors $\text{coh}X \to \text{coh}\mathbb{P}^1$ coming from a double cover $X \to \mathbb{P}^1$. 
Thank you!