A representation theoretic study of noncommutative symmetric algebras (joint with Daniel Chan)

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always work over a field $k$
Part 1

Introduction - The Projective Line
The category $\text{coh} \mathbb{P}(V)$

- Let $V$ is 2-diml.
- Define $\mathbb{P}(V) := \text{Proj } S(V)$.
- Let $\text{coh} \mathbb{P}(V)$ be subcategory of $\text{Qcoh} \mathbb{P}(V)$ consisting of noetherian objects.

### Properties of $\text{coh} \mathbb{P}(V)$

- $\text{coh} \mathbb{P}(V)$ is hereditary.
- Every object of $\text{coh} \mathbb{P}(V)$ is a direct sum of a torsion sheaf and a vector bundle. Every vector bundle is a direct sum of $\mathcal{O}(i)$’s (Grothendieck).
\( V := kx \oplus ky \)

**The Kronecker Algebra \( \Lambda(V) \)**

\[ \Lambda(V) = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix} \]

(Right) \( \Lambda(V) \)-module

\[(N_0, N_1) \text{ and } x, y \in \text{Hom}_k(N_0, N_1) \text{ w/mult} \]

\[
(n_0, n_1) \cdot \begin{pmatrix} a & cx + dy \\ 0 & b \end{pmatrix} := (n_0 a, n_0 cx + n_0 dy + n_1 b)
\]

Notation: \( N_0 \xrightarrow{x} N_1 \)
If $a, b \in k$ not both zero, 

\[
k \xrightarrow{a} k \\
\downarrow{b} \\
k \xrightarrow{c a} k \\
\downarrow{c b} \\
k
\]

is indecomposable. If $c \neq 0$, then isomorphic to

\[
k \xrightarrow{ca} k \\
\downarrow{cb} \\
k
\]
Beilinson’s Theorem

Theorem (Beilinson 1978)

The functor $\text{RHom}(\mathcal{O} \oplus \mathcal{O}(1), -)$ gives an equivalence

$$D^b(\text{coh}\mathbb{P}(V)) \rightarrow D^b(\text{mod}\Lambda(V)).$$

Regular $\Lambda(V)$-Modules

$M$ is regular if it is direct sum of indecomposable $N = (N_0, N_1)$ with $\dim_k(N_0) = \dim_k(N_1)$.

- torsion sheaves over $\mathbb{P}(V)$ $\longleftrightarrow$ regular $\Lambda(V)$-modules
- torsion-free sheaves over $\mathbb{P}(V)$
  $= \{N| \text{Hom}_{\mathbb{P}(V)}(\mathcal{T}, N) = 0 \text{ for all torsion } \mathcal{T}\}$. 
Goal of Talk

Explore the extent to which categories of coherent sheaves over "noncommutative versions" of $\mathbb{P}(V)$ have similar properties, i.e.

- are hereditary,
- have a version of Grothendieck’s Theorem for objects, and
- satisfy a version of Beilinson’s Theorem.
Part 2

Piontkovski’s $\mathbb{P}_n^1$
Coherent rings and modules

A is \( \mathbb{N} \)-graded algebra, \( \text{Gr}A \) cat. of graded right \( A \)-modules.

**Definition**

Suppose \( M \in \text{Gr}A \)

- \( M \) is **coherent** if \( M \) is f.g. and every f.g. submodule is finitely presented.
- \( A \) is coherent if it is coherent as a graded right \( A \)-module.

**Theorem (Chase (1960))**

\( A \) is coherent iff the full subcategory of \( \text{Gr}A \) of coherent modules is abelian.

If \( A \) is coherent, let \( \text{coh}A \) denote this subcategory of \( \text{Gr}A \).
Examples/Nonexamples of (graded right) coherence

- $A$ noetherian $\implies A$ is coherent.
- $k[\{x_i\}_{i \in \mathbb{N}}]$ is coherent.
- $k\langle x_1, \ldots, x_n \rangle$ is coherent.
- $k\langle x, y, z \rangle/\langle xy, yz, xz - zx \rangle$ is not coherent (Polishchuk (2005)).
Let $A$ be coherent connected $\mathbb{N}$-graded algebra and let
- $\text{coh} A =$ cat. of (graded right) coherent modules
- $\text{tors} A =$ full subcat. of right-bounded modules.

**Definition**

$\text{cohproj} A := \text{coh} A / \text{tors} A$

**Remark**

If $A$ is noetherian, $\text{cohproj} A \equiv \text{proj} A$.
Theorem (Zhang (1998))

If $A$ is connected, gen. in degree 1 and regular of dim 2 then

$$A \cong k\langle x_1, \ldots, x_n \rangle / \langle b \rangle$$

where $n \geq 2$, $b = \sum_{i=1}^{n} x_i \sigma(x_{n-i+1})$ and $\sigma \in \text{Aut } k\langle x_1, \ldots, x_n \rangle$. Furthermore, $A$ is noetherian iff $n = 2$.

Theorem (Piontkovski (2008))

$n > 2$ implies $A$ is coherent. If $\mathbb{P}_n^1 := \text{cohproj} A$, then $\mathbb{P}_n^1$ depends only on $n$. Furthermore, $\mathbb{P}_2^1 \equiv \text{cohP}(k^{\oplus 2})$, $\mathbb{P}_n^1$ is hereditary, and there is a Beilinson equivalence

$$D^b(\mathbb{P}_n^1) \equiv D^b(\text{mod}\Lambda(k^{\oplus n})).$$

This form of Beilinson equivalence was independently discovered by Minamoto (2008) and Van den Bergh.
Part 3

Noncommutative Projective Lines
The orbit algebra of a sequence

If $\mathcal{L} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ is a sequence of objects in a category $C$, then

$$(A_{\mathcal{L}})_{ij} = \text{Hom}(\mathcal{L}_{-j}, \mathcal{L}_{-i})$$

with mult. = composition makes $A_{\mathcal{L}} = \bigoplus_{i,j \in \mathbb{Z}} (A_{\mathcal{L}})_{ij}$ a $\mathbb{Z}$-algebra.

A ring $A$ is a (positively graded) $\mathbb{Z}$-algebra if

- There exists a vector space decomposition $A = \bigoplus_{i,j \in \mathbb{Z}} A_{ij}$, with $A_{ij} = 0$ if $j < i$,
- $A_{ij}A_{jk} \subset A_{ik}$,
- $A_{ij}A_{kl} = 0$ for $k \neq j$, and
- the subalgebra $A_{ii}$ contains a unit.
Noncommutative versions of coherent sheaves over $\mathbb{P}^1$: Bimodules

Goal

If $V$ is 2-diml/$k$, $\mathbb{P}^1 = \mathbb{P}(V)$. Idea: replace $V$ by bimod. $M$.

- $D_0, D_1 =$ division rings over $k$
- $M = D_0 - D_1$-bimodule of left-right dimension $(m, n)$

Right dual of $M$

$M^* := \text{Hom}_{D_1}(M_{D_1}, D_1)$ is $D_1 - D_0$-bimodule with action 

$$(a \cdot \psi \cdot b)(x) = a\psi(bx).$$

Can define $^*M = M^{-1*}$ similarly.
Noncommutative versions of coherent sheaves over $\mathbb{P}^1$: Definition (Van den Bergh (2000))

Let $M$ be $D_0 - D_1$-bimodule.

**Definition of $S^{nc}(M)$**

- $\exists \eta_i : D \to M^i \otimes_D M^{i+1}$
- $S^{nc}(M)_{ij} = \frac{M^i \otimes \cdots \otimes M^{-1}}{\text{relns. gen. by } \eta_i}$ for $j > i$,
- mult. induced by $\otimes$.

**Definition of $\mathbb{P}^{nc}(M)$**

Suppose $S^{nc}(M)$ is coherent. We let

$$\mathbb{P}^{nc}(M) := \text{cohproj}S^{nc}(M)$$

**Motivation for our work**

Under what conditions on $M$ is $S^{nc}(M)$ coherent?
Examples

- $\mathbb{P}^1_n \equiv \mathbb{P}^{nc}(k^{\oplus n})$ (N (2017)). Implies $\mathbb{P}^1_n = \text{cohproj}k\langle x_1, \ldots, x_n \rangle/\langle b \rangle$ independent of choice of $b$.
- Noncommutative curves of genus zero after Kussin (N (2015))
- Artin's Conjecture: Every noncommutative surface infinite over its center is birational to some $\mathbb{P}^{nc}(M)$ (1997)
Part 4

Main Theorem
Definition

\( D_0, D_1 \) division rings (with \( \text{char} \neq 2 \)), \( M = D_0 - D_1 \)-bimodule with left-right dimension \((m, n)\). We let

\[
\Lambda(M) := \begin{pmatrix} D_0 & M \\ 0 & D_1 \end{pmatrix}
\]

Definition

Let \( M \) be a \( D_0 - D_1 \)-bimodule. \( M \) has \textit{symmetric duals} if \( M \) and \( M^* \) are finite-dimensional on the left and right and there is a bimodule isomorphism \( M \cong M^{**} \).
Main Theorem: Statement

Hypothesis: $M$ has symmetric duals, and the product of its left and right dimensions is $\geq 4$.

Theorem (Chan-N (2018))

- $\mathbb{S}^{nc}(M)$ is coherent,
- $\mathbb{P}^{nc}(M)$ is hereditary, and
- There is an equivalence $D^b(\mathbb{P}^{nc}(M)) \equiv D^b(\text{mod}\Lambda(M))$.

Definition

$\mathcal{T} \in \mathbb{P}^{nc}(M)$ is *torsion* if it corresponds to a regular module over $\Lambda(M)$. $\mathcal{N} \in \mathbb{P}^{nc}(M)$ is *torsion-free* if $\text{Hom}_{\mathbb{P}^{nc}(M)}(\mathcal{T}, \mathcal{N}) = 0 \ \forall \ \mathcal{T}$.

Theorem (Chan-N (2018))

Every object of $\mathbb{P}^{nc}(M)$ is a direct sum of a torsion sheaf and a torsion-free sheaf. Every torsion-free sheaf over $\mathbb{P}^{nc}(M)$ is a direct sum of sheaves of the form $e_i \mathbb{S}^{nc}(M)$.
Grothendieck’s Theorem
Follows from elementary torsion theory!

Remark
Generalizes results from (N (2014)), (Chan-N (2016)) and simplifies their proofs significantly.
Thank you for your attention!
Main Theorem: Proof

Idea

\( \Lambda(M) \) Artinian and hereditary. Work in \( \text{mod}\Lambda(M) \).

Key Computation

We compute \( \text{RHom}_{D^b(\text{mod}\Lambda(M))}(D(\Lambda(M)),-) \) using bimod right projective resolution of \( D(\Lambda(M)) \) inspired by (Butler-King (1999)).

Proof sketch:

- **Step 1:** In commutative case, \( O(i) \in \text{coh}\mathbb{P}(k^{\oplus 2}) \) corresponds to \( L_i \in D^b(\text{mod}\Lambda(k^{\oplus 2})) \). Now let \( N_i \in D^b(\text{mod}\Lambda(M)) \) denote noncommutative analogue of \( L_i \).

- **Step 2:** Prove \( S^{nc}(M) \cong \bigoplus_{i,j} \text{Hom}_{D^b(\text{mod}\Lambda(M))}(N_{-j},N_{-i}) \).

- **Step 3:** Use theorem of Minamoto (2012) to show \( \{N_i\} \) is ample. The fact that \( S^{nc}(M) \) is coherent follows from (Polishchuk (2005)).