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0. PROLOGUE — ABSOLUTE VALUE INEQUALITIES

0.1 Absolute Value Inequalities

We will spend the quarter estimating the distance between numbers. It seems appropriate, therefore, to begin by reviewing briefly the relevant material about the manipulation of absolute value inequalities. The main point will be this: to solve an inequality like $|2x - 5| < 3$, think of the picture rather than just trying to do algebra. Indeed the secret of success for this entire course is to develop (or use if you already have it) the ability to translate back and forth between diagrams on the number line and in the plane and inequalities. Some algebraic manipulation is unavoidable, but to avoid manipulative errors, geometric insight is essential.

To begin at the beginning, all that follows in this course is based on this familiar fact:

The distance between two real numbers $a$ and $b$ is $|a-b|$.

Of course we could equally well write $|b-a|$, since the two absolute values are identical.

As a first application, consider the inequality $|x - 4| < 3$. The wrong way to solve this is algebraically, by considering separately the two cases $x - 4 \geq 0$ and $x - 4 < 0$. (Though of course you will get the correct answer if you don’t screw up.) But it is much better to use the interpretation above and draw a picture: the inequality means that the distance between $x$ and 4 is less than 3. So the allowable $x$’s go from 3 less than 4 to 3 greater than 4, without the endpoints. Thus, with no algebra, the solution is $1 < x < 7$. See the left diagram below. Note that the analysis would be exactly the same for the inequality $|4 - x| < 3$, since $|4 - x| = |x - 4|$ for all real numbers $x$.

What about $|x + 4| < 3$? Well, as you would expect, the extra wrinkle here is just interpreting the given sum as a difference: we consider $|x + 4| = |x - (-4)| < 3$, note that now $x$ must be within 3 of $-4$ and arrive at $-7 < x < -1$. See the right diagram below.
What about the slightly more complicated inequality $|2x - 5| < 3$ that we began with? Well, we can reduce this to the situation already considered just by dividing the inequality through by 2 to get $|x - 5/2| < 3/2$. Now the numbers within 3/2 of 5/2 are $1 = 5/2 - 3/2 < x < 5/2 + 3/2 = 4$. See the left diagram below.

**Warning:** Remember that multiplying both sides of an inequality by a real number is potentially more hazardous than multiplying both sides of an equation by a number. As you know, multiplying both sides by a negative number reverses the direction of the inequality. But you have to be sure that that is what you are really doing. It would be incorrect, for example, to go from $|2x - 5| < 3$ to $|5 - 2x| > -3$, because the left side of the inequality has not changed at all – the absolute value signs have killed the negative sign.

Absolute value inequalities that go the other way can be handled similarly. To solve $|8 - 3x| > 7$, we divide through by 3 to get $|8/3 - x| > 7/3$ and look for the numbers whose distance from 8/3 is more than 7/3. It is obvious that these are of two kinds – those less than $8/3 - 7/3 = 1/3$ and those greater than $8/3 + 7/3 = 5$. See the right diagram above.

So far I have mentioned only strict inequalities – inequalities involving < signs. But if $\leq$ is substituted for $<$, the only difference is that endpoints are included instead of omitted.

This is all there is to know about linear inequalities. When we get to inequalities involving more complicated functions, we will need a few more wrinkles. The most important of these will be introduced in the next section.

**EXERCISES.**

1. Write as absolute value inequalities, and illustrate with a diagram on the number line.
   (a) the distance between $x$ and $-1$ is less than $1/2$
   (b) $x$ is within $2/3$ of $3/4$
   (c) $s$ is more than $\pi$ from $e$
   (d) the distance between $y$ and $z$ is at most $.83$
   (e) the distance between $u$ and $34$ is at least $10^6$
   (f) $v$ is less than $10^{-6}$ away from $271$. 

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**PROLOGUE – ABSOLUTE VALUE INEQUALITIES**
2. Solve the inequalities. Express the answers both as an inequality (like ... < x < ... or whatever is appropriate), and as a diagram on the number line.
(a) \(|x - 2| < 1\)  
(b) \(|1 - x| < 2\)  
(c) \(|x + 3| \geq 3\)  
(d) \(|2x - 5| < 200\)  
(e) \(\left|\frac{2x + 1}{3}\right| < .01\)  
(f) \(\left|2 - \frac{t}{2}\right| < \frac{1}{2}\)  
(g) \(|8 - 3s| > 9\)  
(h) \(|2t + 5| \geq 1\).

3. Sketch the interval on the number line and express it using an absolute value inequality. (e.g. \(0 < x < 2\) is equivalent to \(|x - 1| < 1\).)
(a) \(3 < x < 9\)  
(b) \(-3 < x < 9\)  
(c) \(3 < x < 8\)  
(d) \(-8 < x < -3\)  
(e) \(-11/2 < x < 4\).

0.2 Working with Absolute Value Inequalities

Very often in these notes we will need to estimate the size of some function \(f(x)\), say \(f(x) = 7 \sin 13x - 4x^3 - 23\), on some interval, say \(-4 \leq x \leq 9\). By “estimate” here I mean to find a positive number \(M\) such that \(|f(x)| \leq M\) for all \(x\) such that \(-4 \leq x \leq 9\) (or whatever the interval is). One way to do this would be to use calculus (or a graphing calculator) to find the extreme values of \(f\) on the interval. This would provide the smallest possible value for \(M\). But we generally don’t need to have the best possible value for \(M\); any value for \(M\) will do as long as it is genuinely greater than \(|f(x)|\) throughout the interval. Thus we will always aim at using the least effort rather than obtaining the smallest possible value.

The key to estimating without much effort is to use the basic inequality
\[ |a + b| \leq |a| + |b|, \]
valid for all real numbers \(a\) and \(b\), or the natural extension to more than two numbers:
\[ |a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|. \]
Since this inequality is so important, I will provide a proof. One way to prove the inequality would be to consider all possible cases—that is, all possible patterns of positive and negative values for \(a\) and \(b\), or for \(a_1, a_2, ..., a_n\). The number of cases quickly becomes very large as \(n\) grows so this is not really a very attractive way to proceed. To do better, we must step back a bit and provide a formal definition of \(|x|\).

**Definition 1** For any real number \(x\) the **absolute value** of \(x\) is
\[ |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} = \max\{x, -x\}. \]

The last equality just says that \(|x|\) is the larger of the two numbers \(x\) and \(-x\). Of course this definition is consistent with the geometric point of view of the previous section that \(|x| = |x - 0|\) is the distance between \(x\) and 0.

**WORKING WITH ABSOLUTE VALUE INEQUALITIES**
Remark: It is helpful to keep both the geometric interpretation of absolute value and these more algebraic formulations in mind whenever working with absolute value expressions. Sometimes one will be the most convenient, and sometimes the other.

Now we use the definition to prove the inequality for two numbers. The proof for more than two terms is similar.

**Proposition 2** For any real numbers $a$ and $b$,

$$|a + b| \leq |a| + |b|.$$  

**Proof.** It is clear from the definition that $a \leq |a|$ and $b \leq |b|$ (In each case the inequality will actually be an equality if $a$ (or $b$) is positive, and will be a strict inequality if the number is negative.) Adding, $a + b \leq |a| + |b|$.

Similarly, $-a \leq |a|$ and $-b \leq |b|$ (Now we have equality if $a$ (or $b$) is negative and strict inequality if it is positive). Thus also

$$-(a + b) = (-a) + (-b) \leq |a| + |b|.$$  

Now we can apply the definition of absolute value backwards. We know both $a + b \leq |a| + |b|$ and $-(a + b) \leq |a| + |b|$. The larger of the two numbers on the left is $|a + b|$. We don’t know which one that is, but it doesn’t matter; either way

$$|a + b| = \max \{a + b, -(a + b)\} \leq |a| + |b|$$  

and we have proved the inequality without considering cases.

Now we can use the inequality to answer the question posed at the beginning of the section.

**Example.** Find a real number $M$ so that $|7 \sin 13x - 4x^3 - 23| \leq M$ for $-4 \leq x \leq 9$. This is easy to do for each of the individual terms:

$$|7 \sin 13x| \leq 7 \text{ for all } x$$  

since the sine function is bounded by 1;

$$|-4x^3| \leq 4 \cdot 9^3 = 2916$$  

on the interval $[-4, 9]$ since obviously its greatest absolute value in this interval occurs at $x = 9$; and of course $|23| = 23$ is just a constant. Then we can use our inequality to say that for $-4 \leq x \leq 9$,

$$|7 \sin 13x - 4x^3 + 23| \leq |7 \sin 13x| + |-4x^3| + |-23|$$  

$$\leq 7 + 2916 + 23 = 2946.$$

In this case the estimate is fairly close to best possible; investigation with a TI-92 reveals that the best value for $M$ would be $|f(9)| \approx 2943.83$. In other examples we would be farther off, but we will see that efficient estimates are not often needed for what we do.
Often it is tempting to try to simplify an inequality in a slightly different way. For instance, to find $M$ so that $|3\sin x + x| \leq M$ for $-2 \leq x \leq 0$, we might try to replace $3\sin x$ by the larger quantity $3$ inside the absolute value signs:

$$|3\sin x + x| \leq |3 + x| \leq 3.$$ 

Unfortunately, this is not a true statement for $-2 \leq x \leq 0$. Using your favorite graphing calculator will lead to the conclusion that the maximum value of $|3\sin x + x|$ on this interval occurs at $x \approx -1.91$ and is about 4.739. What has gone wrong is that both $3\sin x$ and $x$ are negative in this interval, so replacing the negative quantity $3\sin x$ by the positive quantity 3 has actually made the absolute value of the sum smaller. In general, you should never make substitutions for part of a sum inside absolute value signs; whether the result is true or not will depend on the signs of the individual terms, something which may be hard to predict and which may even change within the interval.

Instead, to estimate $|3\sin x + x|$ without getting out your graphing calculator, use the inequality from above,

$$|3\sin x + x| \leq |3\sin x| + |x| \leq 3 + 2 = 5.$$ 

This is not as accurate as we would find from actually looking at the graph, but it is much less work.

**EXERCISES.**

1. For each part (a function $f$ and an interval $[a, b]$), find a number $M$ such that $|f(x)| \leq M$ for all $x$ in $[a, b]$ by using the method of this section.
   - (a) $f(x) = 13 - 4\sin 3x + 5x^3$ on $[-2, 1]$
   - (b) $f(x) = e^{-4x} - 14x + 3$ on $[-3, 4]$
   - (c) $f(x) = 13\sin 3x \cos 5x - 3x^4$ on $[-3, 2]$.

2. When does equality occur in the inequality $|a + b| \leq |a| + |b|$? Give one example of a pair of numbers for which there is equality, and another example of a pair of numbers for which the two sides of the inequality are actually different.

3. Use the definition of $|x|$ above to prove that $|a_1 + a_2 + a_3| \leq |a_1| + |a_2| + |a_3|$.

4. Prove that for any real numbers $x$ and $y$, $||x| - |y|| \leq |x - y|$. Suggestion: Imitate the proof of inequality (2) by proving $|x| - |y|$ and its negative less than the right hand side separately. For each part you will find (2) helpful. Explain under what circumstances the inequality is actually an equality, and for which $x$ and $y$ it is a strict inequality.
0.3 Equality

It doesn’t seem to be talked about much, but “=” is used in two different ways in mathematics and science. The distinction between them is very important for this course, so I'll include a couple of paragraphs of explanation.

The first use of the equals sign is the simpler one: when we write, say,

\[ 2 + 3 = 5 \]

we mean that the two sides of the equation have the identical value; they are just written in different forms. Similarly, when we use the rules of algebra to manipulate an expression

\[(x - 2) (x + 2) = x^2 - 4\]

we mean by the equals sign that the two expressions have the identical value no matter what real number is substituted for \(x\). Occasionally we need to be a little careful with this; for

\[\frac{x^2 - 4}{x + 2} = x - 2,\]

the two sides have the same value for all values of \(x\) except \(x = -2\). Ignoring exceptions like this can sometimes lead to trouble.

The second use appears in “equations” like

\[\pi = 3.14.\]

Here we mean not that two numbers are identical, but that they are “close.” This is a much more complicated situation, because one always has the implicit (or better, explicit) question “How close?” That is, whenever there is an error involved, it is usually important to have information about the possible size of the error. And if such an equation is to be manipulated, then it is usually important to keep track of how the possible error can change. This can be difficult, or at least tedious. (Probably many of you have done physics problems in which you keep track of error bounds throughout a calculation.) The rules for manipulating such equations are not exactly the same; for instance, one might also write \(\pi = 3.14159\), but one hardly ever sees \(3.14 = 3.14159\), although the principle that if two quantities are each equal to a third (\(\pi\) here) then they are equal to each other is often used with the simpler meaning of equality. (One could try to give a correct meaning to \(3.14 = 3.14159\) by noting that the usual implicit assumption when replacing a number by a decimal approximation is that the decimal is correct as far as it goes; thus one could interpret this equation as meaning that the result of rounding 3.14159 to two decimal places is 3.14. This is a correct statement, but we would normally express it in a different way, such as the way I have just expressed it.)

Mathematicians, when they are trying to be careful, often distinguish between these two uses of “=” by using a different symbol for the second one.
Since the distinction is crucial for this course, I will try to write $\pi \approx 3.14$ or $\pi \approx 3.14159$ if situations like this arise. Thus “=” in these notes will always mean “identical” rather than “close.”
1. THE RIEMANN INTEGRAL

1.1 The Definition

Many minor variations are possible in the definition of the Riemann integral of a bounded function $f$ on a closed, bounded interval $[a, b]$. Given sufficient patience and technique, any two versions can be shown to be equivalent, that is, a function is integrable with respect to one definition if and only if it is integrable with respect to any other, and the value of the integral is the same for all versions. The version given here is chosen to be as simple as possible.

Definition 3 A real-valued function $f$ defined on an interval $I$ (which may be closed or not, bounded or not) is bounded above if there is a real number $M$ such that

$$f(x) \leq M \text{ for each } x \in I.$$ 

Each such number $M$ is called an upper bound for $f$ on $I$. Similarly, $f$ is bounded below on $I$ if there is a real number $m$ such that

$$f(x) \geq m \text{ for each } x \in I.$$ 

Each such number $m$ is called a lower bound for $f$ on $I$.

The function $f$ is bounded on $I$ if it is both bounded above and bounded below. Equivalently, $f$ is bounded if there is an $M$ such that

$$|f(x)| \leq M \text{ for each } x \in I.$$ 

Example. The function $f(x) = x^2$ is bounded below on any interval. It is bounded above on any bounded interval, such as $[-1, 3]$, but not bounded above on the entire real line, $\mathbb{R}$.

Remark. If $f$ is bounded above on $I$, then $f$ has many possible upper bounds on $I$. For instance, for the function $f(x) = x^2$ on $[-1, 3]$, any number greater than or equal to 9 is an upper bound. In any specific situation the identity of the smallest upper bound is frequently clear (9 here, obviously). It is most often the greatest value of the function on the interval and this is usually the most efficient upper bound to use. Similarly, if $f$ is bounded below, then the greatest of the many lower bounds is frequently obvious (What is it in the example? Be a little careful.), is usually the smallest value of the function on the interval, and is usually the most efficient to use.

Often the situation is a little less obvious than with $f(x) = x^2$. For instance, it is clear that $g(x) = x^2 + 3x - 2$ is bounded below on $\mathbb{R}$ (why?), but it would take a bit of calculation to determine a specific lower bound. An obvious lower
bound for \( f(x) = \sin x + \cos x \) is \(-2\) and an obvious upper bound is \(2\) (why?),
but it would take a little effort to determine that the greatest lower bound is \(-\sqrt{2}\)
and the least upper bound is \(\sqrt{2}\). (Can you do this?) For many purposes
the cruder estimates are adequate. We will see as we go along in this course that
we are constantly trying to estimate upper bounds or lower bounds. Usually
we will be trying to get a sufficiently good estimate with the least possible
effort; part of the skill is determining how good is “sufficiently good.”

**Definition 4** Let \( f \) be a bounded function on a closed, bounded interval \([a, b]\).
Let \( n \) be a positive integer and set \( \Delta = \frac{b-a}{n} \). Then \([a, b]\) can be viewed as being
partitioned into \( n \) subintervals of equal length \( \Delta \). If we set 
\( a_k = a + k\Delta \), then for \( k = 1, \ldots, n \) the \( k \)-th subinterval is \([a_{k-1}], a_k]\).
(Note that \( a_0 = a, a_n = b \).)

An **\( n \)-upper sum** for \( f \) on \([a, b]\) is a sum of the form

\[
U(f, n) = \sum_{k=1}^{n} M_k (a_k - a_{k-1}) = \sum_{k=1}^{n} M_k \Delta = \Delta \sum_{k=1}^{n} M_k
\]

where \( M_k \) is an upper bound for \( f \) on \([a_{k-1}], a_k]\).

An **\( n \)-lower sum** for \( f \) on \([a, b]\) is a sum of the form

\[
L(f, n) = \sum_{k=1}^{n} m_k (a_k - a_{k-1}) = \sum_{k=1}^{n} m_k \Delta = \Delta \sum_{k=1}^{n} m_k
\]

where \( m_k \) is a lower bound for \( f \) on \([a_{k-1}], a_k]\).

**Remark.** Suppose that \( f(x) \geq 0 \) on \([a, b]\). Then we are accustomed to
thinking of \( \int_{a}^{b} f(x) \, dx \) as representing the area bounded by the graph of \( f \)
and the \( x \)-axis between the vertical lines \( x = a \) and \( x = b \). The sums in Definition
4 should be thought of as the sum of the areas of \( n \) rectangles with common
width \( \Delta \) which either enclose the area bounded by the graph of \( f \) completely
(an upper sum) or are completely contained within this area (a lower sum).
Thus we expect all such sums to satisfy the inequalities

\[
L(f, n) \leq \int_{a}^{b} f(x) \, dx \leq U(f, n).
\]
In fact these inequalities are not dependent on the values of $f$ being non-negative. As the diagrams show, they remain true even when $f$ takes on negative values for some or all of the interval $[a, b]$. We just have to remember the usual interpretation in terms of signed area.

The idea of the formal definition of the definite integral will be to turn inequality (1.1) around and define the number $\int_a^b f(x) \, dx$ to be the one trapped in this way between all upper and all lower sums whenever this procedure defines a number unambiguously.

Before stating the formal definition, however, let’s investigate how much “room” there can be between an upper sum and a lower sum. For simplicity we consider for the moment only non-decreasing functions, that is, functions $f$ with the property that $x_1 \geq x_2$ implies $f(x_1) \geq f(x_2)$. Notice that then a left-hand sum is a lower sum and a right-hand sum is an upper sum, that is, we can take $m_k = f(a_{k-1}), M_k = f(a_k)$.

**Example.** The first function illustrated above is $f(x) = 10 + x^3$ on the interval $[-2, 4]$. If we divide this into two subintervals, each of length 3, we find

$$U(f, 2) = f(1) \cdot 3 + f(4) \cdot 3$$

$$= 11 \cdot 3 + 74 \cdot 3 = 255$$

and

$$L(f, 2) = f(-2) \cdot 3 + f(1) \cdot 3$$

$$= 2 \cdot 3 + 11 \cdot 3 = 39.$$  

These are not particularly close, in fact $U(f, 2) - L(f, 2) = 255 - 39 = 216$. We can get them much closer by using $n = 12$ as in the diagram above. Then for $j = 1, 2, ..., 12$ the $j$-th rectangle from the left has base $\frac{j}{3}$ and extends from $-2 + (j - 1)/2$ to $-2 + j/2$. (For instance, the right hand end of the 12-th
rectangle is \(-2 + 12/2 = 4\). We get

\[
U(f, 12) = \sum_{j=1}^{12} \left( 10 + \left( -2 + \frac{j}{2} \right)^3 \right) \cdot \frac{1}{2} = 138.75
\]

\[
L(f, 12) = \sum_{j=1}^{12} \left( 10 + \left( -2 + \frac{j-1}{2} \right)^3 \right) \cdot \frac{1}{2} = 102.75
\]

and now \(U(f, 12) - L(f, 12) = 36\).

You might notice from the Example that when we used 6 times as many intervals, the difference \(U(f, n) - L(f, n)\) was divided by 6. This suggests that there may be a simple relationship between \(n\) and \(U(f, n) - L(f, n)\), at least for non-decreasing functions. To test this, double the number of subintervals again so that we now have 24 subintervals of length 1/4. The result is

\[
U(f, 24) = \sum_{j=1}^{24} \left( 10 + \left( -2 + \frac{j}{4} \right)^3 \right) \cdot \frac{1}{4} = 129.1875
\]

\[
L(f, 24) = \sum_{j=1}^{24} \left( 10 + \left( -2 + \frac{j-1}{4} \right)^3 \right) \cdot \frac{1}{4} = 111.1875.
\]

Now the difference is \(U(f, 24) - L(f, 24) = 18\), just half as much as with 12 intervals. If we draw a diagram with twelve subintervals, but include the boxes for both upper and lower sums, we can see why this is.

For each subinterval, the shaded region is the part of the area of the “upper sum rectangle” that is not in the “lower sum rectangle.” These “error rectangles” are neatly arranged so that the top of any error rectangle is exactly even with the bottom of the next error rectangle to the right. This is just a consequence of the function being non-decreasing; the largest value of the function in any subinterval occurs at the right hand end of the subinterval and so is also the smallest value of the function in the next subinterval to the right.
$U(f, 12) - L(f, 12)$ is just the sum of the areas of all these error rectangles. Since they fit together vertically, we can sum the areas even without knowing the individual areas by pushing all the error rectangles horizontally so that they form one high rectangle of width $1/2$ and height extending from the bottom of the left hand error rectangle, $f(-2) = 2$, to the top of the right hand error rectangle, $f(4) = 74$. This makes the total area

$$(74 - 2) \cdot \frac{1}{2} = 72 \cdot \frac{1}{2} = 36$$

which is what we got before.

But now we see that this method can be used to compute $U(f, n) - L(f, n)$ for any integer value of $n$. If there are $n$ subintervals, each will have length $6/n$. A diagram of $U(f, n) - L(f, n)$ corresponding to the one above for $n = 12$ will have $n$ error rectangles, each of width $6/n$ which fit together in the same way, that is with the top of each error rectangle even with the bottom of the next error rectangle to the right. The bottom of the left hand error rectangle is still at height $f(-2) = 2$ and the top of the right hand error rectangle is still at $f(4) = 74$. Thus we can push the $n$ error rectangles together into a single rectangle with the same height of 72 and width $6/n$. We conclude that for each positive integer $n$,

$$U(f, n) - L(f, n) = 72 \cdot \frac{6}{n} = \frac{432}{n}.$$ 

We can use this how many intervals it would take to get the upper and lower sums to agree to within any specified tolerance. For instance, If we want $U(f, n) - L(f, n) < .01$ then we must have

$$\frac{432}{n} < .01 \text{ or } n > \frac{432}{.01} = 43200.$$ 

That is a lot of subintervals! This is not really a very efficient way to estimate areas or integrals, but that is not the objective here. Remember that we are investigating the difference $U(f, n) - L(f, n)$ because we want to define $\int_a^b f(x) \, dx$ to be the number trapped above all lower sums and below all upper sums, and this will be a satisfactory definition only if there is only one such number. Thus the importance of the formula $U(f, n) - L(f, n) = \frac{432}{n}$ is not that it gives a practical method for estimating $I = \int_{-2}^4 (10 + x^3) \, dx$ but that it does assure us that the upper and lower sums can be made arbitrarily close to one another just by taking enough subintervals. This in turn implies that only one number can be less than all upper sums and more than all lower sums, because if there were two such numbers, say $I_1$ and $I_2$, then they would be a certain positive distance apart, $|I_1 - I_2|$ in fact, and differences $U(f, n) - L(f, n)$ would have to be at least that big. But this would contradict the fact that $U(f, n) - L(f, n) = \frac{432}{n}$ can be made smaller than any positive number smaller than any positive.
number (such as $|I_1 - I_2|$) just by making $n$ big enough. Thus there is only one number trapped between the lower sums and the upper sums. In this particular case we can use standard calculus methods to discover that the number is 120.

The previous example shows that for the function $f(x) = 10 + x^3$, looking at the difference between upper and lower sums is a way to see that there is only one possible value for $\int_{-2}^{4} (10 + x^3) \, dx$. Thus encouraged, we adopt the following general definition.

**Definition 5** A bounded function $f$ (no longer necessarily non-decreasing) defined on a closed, bounded interval $[a, b]$ is **Riemann integrable** on $[a, b]$ if for each $\epsilon > 0$, there is a positive integer $n$ (depending on $\epsilon$) such that

$$U(f, n) - L(f, n) < \epsilon. \tag{1.2}$$

If $f$ is Riemann integrable on $[a, b]$, we denote the unique real number trapped between all upper sums and all lower sums (as in inequality (1.1)) by

$$\int_a^b f(x) \, dx.$$

**Remark.** Note that the Riemann integral is defined only for bounded functions, since otherwise $U(f, n)$ and $L(f, n)$ do not make sense. We will see later that the integral of some unbounded functions can be defined as the limit of the integrals of a sequence of bounded functions. (One type of improper integral.)

**Examples.**

1. To use the definition to show that $10 + x^3$ is Riemann integrable on $[-2, 4]$, we must be able for any tolerance $\epsilon$ to find $n$ so that $U(f, n) - L(f, n) < \epsilon$. We know from above that $U(f, n) - L(f, n) = \frac{432}{n}$ so we must solve the inequality $\frac{432}{n} < \epsilon$ or equivalently $\frac{432}{\epsilon} < n$, that is, we can choose $n$ to be any integer greater than $\frac{432}{\epsilon}$.

2. Exercise 7 establishes a formula in terms of $n$ for $U(f, n) - L(f, n)$ when $f(x) = x^2 + 1$ on $[0, 3]$, and Exercise 8 asks you to use that formula in the same way to find a similar inequality relating $n$ to $\epsilon$.

3. Exercise 9 establishes a similar formula for $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 2 \\ x^2 + 2 & \text{if } 2 < x \leq 3 \end{cases}$ on $[0, 3]$. Thus functions with jumps in their graphs can be Riemann integrable according to this definition.

4. The first example of a bounded function defined on $[0, 1]$ that is not Riemann integrable there was given by P. Lejeune Dirichlet in 1829. It is

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

For this function and any positive integer $n$, $U(f, n) \geq 1$, and $L(f, n) \leq 0$. (Why? Be sure you understand this.) Notice that it would be impossible to
draw the graph of this function in the usual sense. We will see that any "reasonable" bounded function does turn out to be Riemann integrable. However part of the importance of this example is that it is the earliest case known to me of a mathematician realizing that it is important to distinguish between "any function" and "any reasonable function" and then stating precisely a class of functions for which a result is true. (His class was much the same as ours will be.) In this sense, Dirichlet’s work on integration has a claim to be considered the first “modern” work on mathematical analysis (roughly the branch of mathematics arising from calculus and the concept of limit.) It took most of the rest of the nineteenth century for this idea that one should specify precisely for what objects a theorem is true to be carried through for all of analysis.

EXERCISES.

1. For \( f(x) = x^2 + 1 \) on \([0, 3]\) compute \( U(f,n), L(f,n) \) and \( U(f,n) - L(f,n) \) for \( n = 3, n = 6, n = 30, n = 60 \) using the greatest (resp. least) value of \( f \) on the \( k \)-th subinterval for \( M_k \) (resp. \( m_k \)). Draw a diagram in each of the first two cases illustrating \( U \) and \( L \). (Use a programmable calculator for \( n = 30 \) and \( n = 60 \). Don’t try it by hand!)

2. Repeat #1 for \( f(x) = x^2 - 2 \) on the interval \([0, 3]\).

3. Repeat #1 for \( f(x) = 3 + 2 \sin x \) on the interval \([0, \pi/2]\).

4. How many subintervals would you need in #1 to have \( U(f,n) - L(f,n) = 0.1 \)? Verify that your result is correct by using your calculator to actually compute the difference for your value of \( n \).

5. A car starts from rest and accelerates. Its speed in feet/second is measured at one second intervals and found to be as in the table. Find an upper estimate and a lower estimate for the distance the car travels in the first 4 seconds. How often would you have to measure the car’s speed in order for the two estimates to be within one foot of one another?

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed</td>
<td>0</td>
<td>20</td>
<td>35</td>
<td>45</td>
<td>50</td>
</tr>
</tbody>
</table>

6. Is it possible to choose \( n \) so that \( U(f,n) - L(f,n) = 0.1 \) for the function and interval of #2? How do you know?

7. Generalize the results of #1 to the case of \( n \) subintervals, that is, guess and then justify a general formula \( U(f,n) - L(f,n) = \) (some function of \( n \)) assuming that \( f, M_k, m_k \) are chosen as in #1. (Suggestions: To compute \( U - L \) you do not have to compute \( U \) and \( L \) separately. Think about which points you evaluate \( f \) at to make \( U \) and \( L \). Look carefully at the examples in #1.)

8. Use the result of #7 to answer #4 again. How many subintervals in order to have \( U(f,n) - L(f,n) < 0.01, < 0.00258, < \epsilon \) (an arbitrary positive real number) Use words and an appropriate diagram to explain your answer.

9. Let \( f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 2 \\ x^2 + 2 & \text{if } 2 < x \leq 3 \end{cases} \). Find a general formula for \( U(f,n) - L(f,n) \) in terms of \( n \) on the interval \([0,3]\). Notice that the jump
in values of $f$ does not really affect the nature of the formula. Now use this to establish an inequality relating $n$ to the tolerance $\epsilon$ from the definition of Riemann integrability.

**Remark.** It is traditional in mathematics to use $\epsilon$ (the lower case Greek letter epsilon, not some form of the Latin letter $e$) to denote a number being used as a tolerance — a difference between two quantities that is not to be exceeded.

### 1.2 The Integral of a Piecewise Monotonic Function

**Definition 6** A function $f$ defined on an interval $I$ is **monotonic** if it is either non-decreasing or non-increasing ($x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$.)

**Theorem 7** A bounded monotonic function $f$ defined on a closed bounded interval $[a, b]$ is Riemann integrable on $[a, b]$.

**Proof.** Left as an exercise.

**Remark.** It is important to realize that a monotonic function may have a jump in its values (or even many jumps) like the function

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 2 \\ x^2 + 2 & \text{if } 2 < x \leq 3. \end{cases}$$

of the previous section.

Theorem 7 is not all that satisfactory, because most functions are not monotonic. (Think, for instance, of any polynomial of degree greater than one, or sin or cos.) Most functions we meet in practice, though, do have the property that they change direction only a finite number of times in any bounded interval $[a, b]$. It is not hard to extend our theorem to this situation, which will be sufficiently general for our purposes. First, a more precise definition of the class of functions.

**Definition 8** A function $f$ defined on a closed, bounded interval $[a, b]$ is said to be **piecewise monotonic** on $[a, b]$ if there is a finite collection of points

$$a = u_0 < u_1 < \ldots < u_{\ell-1} < u_{\ell} = b$$

such that $f$ is monotonic on each subinterval $(u_{j-1}, u_j)$, $j = 1, 2, \ldots, \ell$. 

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THE Riemann INTEGRAL
The idea is that \( f \) changes direction from increasing to decreasing or the reverse at each of the points \( u_j \) for \( j = 1, 2, \ldots, n-1 \) for a total of \( \ell-1 \) direction changes. \( u_0 \) and \( u_\ell \) are the endpoints of the interval.

**Example.** The function \( \cos x \) changes direction at each multiple of \( \pi \), so on any interval \([a, b]\) such that neither \( a \) nor \( b \) is a multiple of \( \pi \), the number of direction changes is equal to the number of multiples of \( \pi \) contained in the open interval. The collection \( \{u_k\} \) will have \( \ell \) equal to one more than the number of multiples of \( \pi \) contained in the open interval \((a, b)\). (In particular, the number of subintervals needed depends on \([a, b]\) as well as \( f \).)

To find \( U(f, n) \) and \( L(f, n) \) for \( f(x) = \cos x \) on the interval \([-1, 1]\) we must be a little more careful than before, since the maximum value of \( \cos x \) will occur in different places in the subinterval depending on which subinterval we are looking at.

For \( n = 2 \) we have
\[
U(f, 2) = f(0) \cdot 1 + f(0) \cdot 1 = 2 \\
L(f, 2) = f(-1) \cdot 1 + f(1) \cdot 1 \approx 1.08.
\]

Here the maximum value of \( \cos x \) in each subinterval occurs at 0 and the minimum occurs at the other endpoint.

For \( n = 3 \), the maximum value of \( \cos x \) on the middle interval occurs in the middle of the interval and not at either endpoint. This is of course just a consequence of the fact that it is in this subinterval that the graph changes from increasing to decreasing.

In neither case does the formula of the preceding section give the correct value of \( U(\cos x, n) - L(\cos x, n) \). In fact since \( \cos(-1) = \cos 1 \), that formula will always give the value 0, which is clearly not correct.

**Remark.** The preceding example illustrates the fact that the situation has, unfortunately become a little more complicated. In general we cannot assume
that the endpoints $x_k$ of the subintervals determined by $n$ fall on the endpoints $u_j$ dividing $[a, b]$ into subintervals where $f$ is monotonic. This means that there will now be individual subintervals $[x_{k-1}, x_k]$ on which $f$ is not monotonic. (The subinterval containing zero in the example; in general any subinterval containing a $u_j$ in its interior.) For such subintervals, either the maximum or minimum value will fall somewhere in the interior of the subinterval, and although the other one falls at an endpoint, which endpoint depends on the individual details of the function and the subinterval. (I am assuming here that $n$ is large enough so that any subinterval $[x_{k-1}, x_k]$ contains at most one of the $u_j$s. Otherwise both the maximum and minimum might occur in the interior.)

It turns out that for piecewise monotonic functions there is no longer a convenient exact formula for $U(f, n) - L(f, n)$. It is, however, still possible to estimate $U(f, n) - L(f, n)$ in terms of $n$, the maximum and minimum values of $f$ on $[a, b]$ (which may no longer be at $a$ or $b$) and the number $\ell$ of subintervals that we need to divide $[a, b]$ into so that $f$ is monotonic on each one. We will be able to use this to prove the following theorem.

**Theorem 9** A bounded piecewise monotonic function $f$ defined on a closed bounded interval $[a, b]$ is Riemann integrable on $[a, b]$.

**Proof.** If $f$ is piecewise monotonic on $[a, b]$, there is a fixed set of points $u_0 = a < u_1 < \ldots < u_\ell = b$ so that $f$ is monotonic on each. Now let $n$ be any positive integer large enough so that only one of the $u_j$s is contained in any one subinterval of length $(b - a)/n$. We can divide the $n$ subintervals into at most $\ell$ groups as follows. First, there are at most $\ell - 1$ subintervals containing a $u_j$ inside it (not at an endpoint). Call these $u$-intervals.

A $u$-interval has the property that either the maximum value or the minimum value of $f$ on the interval occurs in the middle rather than at one endpoint. (For instance for the situation of Exercise 3 below where $[-1, 2]$ is divided into 5 subintervals, we have $u_0 = -1, u_1 = 0, u_2 = 2$ (so $\ell = 2$ here) and there is one $u$-interval namely the interval $[-2/5, 1/5]$ containing 0.) However if the minimum value occurs inside the $u$-interval, then the maximum occurs at one of the endpoints (at $-2/5$ in the situation of Exercise 3) and if the maximum occurs inside then the minimum occurs at one of the endpoints.

The $u$-intervals divide the other subintervals into at most $\ell$ regions on each of which $f$ is monotonic. (In Exercise 3 there are two such groups—the single interval $[-1, -2/5]$ is one group where $f$ is decreasing and the three intervals $[1/5, 4/5], [4/5, 7/5], \text{ and } [7/5, 2]$ form another group where $f$ is increasing.) Moreover, if we add each $u$-interval to the group on the side of the endpoint where $f$ has either its maximum or its minimum (the left side in Example 3 since the maximum is at $-2/5$), then the set of error rectangles in each group do not overlap vertically so that we can estimate their total area by the stacking method we used for monotonic functions.

Now the terms in $U(f, n) - L(f, n)$ corresponding to the intervals in any single group sum to at most $(M - m) \frac{b-a}{n}$ by the same reasoning as for monotonic
functions—the error rectangles for any group can be stacked into a single rectangle of total height at most \( M - m \) and with width \( \frac{b - a}{n} \). Now adding the estimates for each group together, we get

\[
U(f, n) - L(f, n) \leq \ell(M - m) \frac{b - a}{n}.
\]

The right hand side of this inequality still has the form \( Kn \), where the constant \( K \) is rather complicated but does not depend on the number of subintervals. Thus we can now proceed as with monotonic functions. To show that \( f \) is Riemann integrable on \([a, b]\) we must for any given tolerance \( \epsilon \) find an integer \( n \) so that \( U(f, n) - L(f, n) < \epsilon \). Since \( U(f, n) - L(f, n) \leq \frac{K}{n} \), it is enough for \( \frac{K}{n} < \epsilon \), and that will be true for any integer \( n > \frac{K}{\epsilon} \). This completes the proof that a piecewise monotonic function is Riemann integrable.

**Example.** Here is a diagram of the difference between \( U(f, 12) \) and \( L(f, 12) \) for the function \( f(x) = 47 - 9x - 12x^2 + 4x^3 \) on \([-2, 4]\). Notice that the function changes direction twice in the interval \([-2, 4]\) and that the “error rectangles” can be divided into three groups that don’t overlap vertically by pushing both of the transitional rectangles into the group in the middle.

**EXERCISES**

1. Prove Theorem 7. Do the case of non-increasing \( f \) in detail and indicate the modifications, if any, for non-decreasing \( f \). (Suggestion: In order to show that \( f \) is Riemann integrable, you must show that for any given positive \( \epsilon \) an integer \( n \) can be found so that inequality (1.2) is satisfied. In this case you can do this by producing an explicit function giving \( n \) in terms of \( \epsilon \) just as in the discussion of the specific function \( 10 + x^3 \) of the preceding section. As in the previous exercises, start by expressing the left side of (1.2) in terms of \( n \).)

2. Find an upper sum and a lower sum for \( f(x) = x^2 \) on \([-1, 1]\) with \( n = 2 \), and \( n = 3 \).

3. Find an upper sum and a lower sum for \( f(x) = x^2 \) on \([-1, 2]\) with \( n = 5 \).
4. Let \( f \) be a bounded function on \([a, b]\), say \( m \leq f(x) \leq M \) for all \( a \leq x \leq b \). Let \( n \) be a positive integer, and let \([a, b]\) be partitioned into \( n \) subintervals of equal length. Explain, using a diagram, why if we group any one term from \( U(f, n) \) with the term from \( L(f, n) \) corresponding to the same subinterval, that term of \( U(f, n) - L(f, n) \) is bounded above by \((M - m) \cdot \frac{b - a}{n}\).

(Suggestion: The term of \( U(f, n) - L(f, n) \) is the area of one rectangle. Interpret the quantity above as the area of a rectangle with the same sides but a possible lower base and higher top.)

5. Work through the proof of Theorem 9 for \( f(x) = \cos x \) on \([-1, 1]\). You should need to partition the subintervals into at most two groups.

### 1.3 A Few Properties of Definite Integrals

It is not the purpose of these notes to develop the familiar properties of definite integrals in a systematic way. However a few properties will be stated here to be available for future use, and the fundamental result on comparing integrals will be proved.

**Theorem 10** A constant function \( f(x) = c \) is Riemann integrable on any interval \([a, b]\) with value \( \int_a^b c \, dx = c(b - a) \).

**Proof.** For any \( n \), \( U(f, n) = L(f, n) = \sum_{k=1}^{n} c \Delta x = c \sum_{k=1}^{n} \Delta x = c(b - a) \).

Thus there is certainly only one number between the upper and lower sums.

**Theorem 11** If \( f \) and \( g \) are Riemann integrable on \([a, b]\), then so is \( f + g \) and

\[
\int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
\]

If \( f \) is Riemann integrable on \([a, b]\) and \( c \) is any real number, then \( cf \) is Riemann integrable on \([a, b]\), and

\[
\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx.
\]

**Theorem 12** If \( f \) and \( g \) are Riemann integrable on \([a, b]\), and if \( f(x) \leq g(x) \) for each \( x \) in \([a, b]\), then

\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx.
\]
Remark For positive functions $f$ and $g$ this just says that the area under the higher graph is greater than the area under the lower graph.

Proof. For any $n$ and any subinterval $[a_{k-1}, a_k]$, if $m_k \leq f(x)$ for all $x$ in $[a_{k-1}, a_k]$, then also $m_k \leq g(x)$ for all $x$ in $[a_{k-1}, a_k]$. Thus $L(f, n) \leq L(g, n) \leq \int_a^b g(x) \, dx$ for each $n$. Since $\int_a^b f(x) \, dx$ is the smallest number greater than or equal to all the numbers $L(f, n)$, it must also be the case that $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$.

**Corollary 13** If $f$ is Riemann integrable on $[a, b]$ and if $m$ and $M$ are real numbers such that

$$m \leq f(x) \leq M$$

for each $x$ in $[a, b]$, then

$$m (b - a) \leq \int_a^b f(x) \, dx \leq M (b - a).$$

Remark. Intuitively, this says that the area under the graph of $f$ is greater than the area of the rectangle whose height is the minimum value of $f$ in $[a, b]$ and less than the area of the rectangle whose height is the maximum value of $f$ on $[a, b]$. (Strictly speaking, $m$ might be less than the minimum value of $f$ and $M$ might be more than the maximum value of $f$.)

Proof. Each individual inequality is just the special case of Theorem 12 where one of the two functions is a constant function. (For the left hand inequality the function $f$ of the Corollary plays the role of $g$ in Theorem 12.)

**Corollary 14** If $f$ is Riemann integrable on $[a, b]$, then

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$
Proof. Since $f(x) \leq |f(x)|$, it follows from Theorem 12 that $\int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx$. Similarly, since $-f(x) \leq |f(x)|$, $-\int_a^b f(x) \, dx = \int_a^b -f(x) \, dx \leq \int_a^b |f(x)| \, dx$. Now either $\left|\int_a^b f(x) \, dx\right| = \int_a^b f(x) \, dx$ or $\left|\int_a^b f(x) \, dx\right| = -\int_a^b f(x) \, dx$, depending on whether $\int_a^b f(x) \, dx$ is positive or negative. Either way, the result of the Corollary is covered by one of the alternatives.

These inequalities can be very useful to study the size of a function whenever we can express it as an integral. Here is a simple example, using the familiar fact that

$$\ln x = \int_1^x \frac{1}{t} \, dt$$

for any $x > 0$.

For any positive integer $n$,

$$\ln \left(1 + \frac{1}{n}\right) = \ln \left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n = \int_1^{n+1} \frac{1}{t} \, dt - \int_1^n \frac{1}{t} \, dt = \int_n^{n+1} \frac{1}{t} \, dt .$$

For $n \leq t \leq n + 1$, $\frac{1}{n+1} \leq \frac{1}{t} \leq \frac{1}{n}$. Thus, comparing areas,

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{1}{t} \, dt = \ln \left(1 + \frac{1}{n}\right) \leq \frac{1}{n}$$

as in this diagram:

We will find this technique of representing a function as an integral in order to estimate its size useful from time to time. According to the Fundamental
Theorem of Calculus we can do this for any function with a reasonable derivative. Here, for reference, is a form of the Fundamental Theorem phrased in the way that we will use it.

**Theorem 15** Let \( f \) be a function with a continuous derivative \( f' \) on some interval containing the point \( a \). Then for any \( x \) in this interval (including values of \( x \) less than \( a \))

\[
f(x) - f(a) = \int_a^x f'(t) \, dt.
\]

**EXERCISES.**

1. Use the integral representation of the natural logarithm to show that for any \( x > 0 \),

\[
\frac{x}{1+x} < \ln(1+x) < x.
\]
Illustrate with a suitable diagram!

2. Show that for \( -1 < x < 0 \),

\[
\left| \frac{x}{1+x} \right| > |\ln(1+x)| > |x|.
\]
You will need to use the fact that if \( b < a \), then \( \int_b^a f(t) \, dt = - \int_a^b f(t) \, dt \). What happens to the inequality if the absolute values signs are removed?

3. Show that \( \frac{1}{2\sqrt{x+1}} < \sqrt{x+1} - \sqrt{x} < \frac{1}{2\sqrt{x}} \) for all real numbers \( x > 0 \) by writing the middle quantity as a definite integral from \( x \) to \( x+1 \). (Use Theorem 15 with an appropriate choice of \( f \) and \( f' \).)

4. Find reasonable upper and lower bounds for \( \sqrt{2n} - \sqrt{2n-3} \), where \( n \) is an integer \( \geq 2 \), by expressing this quantity as a definite integral.

5. Use an integral representation to show that for all \( x \geq 0 \), \( \sqrt{x^2+1} - x \leq \frac{1}{2x} \). (Hint: \( x = \sqrt{x^2} \).)
2. LIMITS OF FUNCTIONS

2.1 Limits at Infinity

We turn now to considering the limit of a real-valued function at a point. For reasons I’ll explain as we go along, we will first consider the concept of limit at the “point” “infinity.”

Example 1. Consider the graph of the function \( f(x) = \frac{x}{x+1} \) for \( x \geq 0 \). (This means draw the graph before reading farther. If you want to let a graphing calculator do the work for you, consider the graph successively on the intervals \([0, 10], [0, 100], [0, 1000]\) with the same \( y \) scale each time (you choose this) to get a feel for what the graph looks like as you view it from “a greater distance.” (Think of hanging in a balloon over the graph and discarding weight so that you go higher and higher and see more and more of the graph. (As you may have noticed, I have a weakness for nested parentheses.)))

What do you see happening as \( x \) is allowed to “go to infinity”? Well, the value of the fraction \( \frac{x}{x+1} \) approaches \( 1 \). Does it ever equal \( 1 \)? (The answer to this question, trivial as it seems, is fundamental to understanding limits. Note also that we are using the word ‘equal’ here in the mathematician’s sense, as in ‘2 + 2 equals 4’, not in the scientist’s sense of ‘the acceleration due to gravity at the earth’s surface equals 9.80 m/sec\(^2\)’. In the second sentence, the meaning of ‘equals’ is something like “These quantities are close enough together for us not to need to distinguish between them in our present circumstances.” This is a well-accepted use of the word ‘equals’ but is not the meaning of the word that will be useful to us in our discussion of limits.)

What does it mean to say that \( \frac{x}{x+1} \) approaches the value 1 as \( x \) gets larger and larger? It is not just that \( \frac{x}{x+1} \) gets closer and closer to 1 as \( x \) gets larger. After all, \( \frac{x}{x+1} \) also gets closer and closer to 2 as \( x \) gets larger. The point is that we have to look at the difference between \( \frac{x}{x+1} \) and 1 (or the difference between \( \frac{x}{x+1} \) and 2). In the first case, this difference \( 1 - \frac{x}{x+1} = \frac{1}{x+1} \) approaches 0; in the second case \( 2 - \frac{x}{x+1} = \frac{x+2}{x+1} \) it does not. Mathematicians struggled for centuries to pin down that phrase ‘approaches zero.’ What they eventually came up with in the nineteenth century was to insist (roughly speaking) that a quantity pass a succession of tests, each specifying a closer approach to 0 than the previous one. Formally, this can be
stated as follows.

**Definition 16** Let $f$ be a function defined for all $x \geq x_0$. Then we say that $f(x)$ has the limit $L$ as $x$ approaches $\infty$ and write

$$\lim_{x \to \infty} f(x) = L$$

if for each $\epsilon > 0$ there is a number $X(\epsilon)$ such that $|L - f(x)| < \epsilon$ whenever $x > X(\epsilon)$.

**Remark.** One should think of this definition in the following terms. The positive number $\epsilon$ specifies a tolerance—a maximum amount that $f(x)$ is allowed to deviate from $L$. Graphically, it defines a horizontal band of width $2\epsilon$ centered at $y = L$ (that is, extending from $y = L - \epsilon$ to $y = L + \epsilon$).

In order for the limit to exist, the graph of $f$ must lie within this band from some point on as $x$ gets larger. This point is the one defined by the number $X(\epsilon)$. Here the function-like notation has the purpose of emphasizing that the choice of $X(\epsilon)$ depends upon the tolerance $\epsilon$ that must be satisfied. It does not imply that there is only one possible choice for $X(\epsilon)$. In the diagram just below the choice for $X(\epsilon)$ could have been a little smaller than the one illustrated, and could also have been larger.

The critical point about the definition is that no one of these tests (that is, no one value of $\epsilon$) suffices to show that $f(x)$ approaches $L$. One test can only show that the quantity $|L - f(x)|$ is less than some single positive tolerance, and it might never get appreciably closer to 0 than that. More generally, a finite number of tests can only show that the quantity is less than the smallest (but still positive) tolerance specified. To show that a quantity approaches 0 it is essential to show that it passes an infinite number of tests. If an object is thinner than $10^{-6}$ cm, it is very thin, but there are real objects (with positive width) that are that thin. If an object must be thinner than each of $10^{-1}$ cm,
10^{-2} \text{cm}, 10^{-3} \text{cm}, \ldots$, where the dots indicate that we just keep going forever with a succession of smaller tolerances, then its width must actually be 0, since if it had any positive width, no matter how small, eventually in that succession of tests we will ask it to meet a tolerance smaller than its width.

This is, of course, reflected in the definition by the insistence that something be true for each $\epsilon > 0$. In practice this means that to show that a limit exists, one must come up with a procedure for showing how to pass any test. It is often helpful to fix ideas by practicing with a specific single choice of $\epsilon$, but the final product must be a general procedure that will work for any positive $\epsilon$.

**Example 1 continued.** Let's see how this works out in showing that

$$
\lim_{x \to \infty} \frac{x}{x + 1} = 1.
$$

We'll practice by finding out when $\frac{x}{x + 1}$ is within $\frac{1}{100}$ of 1, that is, when

$$
\left| 1 - \frac{x}{x + 1} \right| < .01. 
$$

Fortunately here it is easy to get rid of the absolute value signs. We have

$$
\left| 1 - \frac{x}{x + 1} \right| = \left| \frac{1}{x + 1} \right| = \frac{1}{x + 1}
$$

so we want to know when $\frac{1}{x + 1} < .01$. It is easy to solve this inequality for $x$ to get $x + 1 > 100$ (note that we have assumed $x + 1 > 0$ by not reversing the direction of the inequality) or $x > 99$.

To try once more, when is $\left| 1 - \frac{x}{x + 1} \right| < .0001$? Well, now we must have $\frac{1}{x + 1} < .0001$ or $x + 1 > 10000$, or $x > 9999$. Not surprisingly, with a stiffer condition to meet, we must go farther out to the right before it becomes satisfied. In general the set of $x'$s that pass the test depends on what the test is.
Now let’s try the real thing. We must find a general procedure that tells us when \( \frac{x}{x+1} \) will be within any specified tolerance of 1. However it is no more difficult. Now we want to know when

\[
1 - \frac{x}{x+1} = \frac{1}{x+1} < \epsilon.
\]

We can again solve for \( x \) to get \( x + 1 > \frac{1}{\epsilon} \) or \( x > \frac{1}{\epsilon} - 1 \). Thus if we set \( X(\epsilon) = \frac{1}{\epsilon} - 1 \), then \( 1 - \frac{x}{x+1} < \epsilon \) for all \( x > X(\epsilon) \).

The calculation we have just made tells us how to choose \( X(\epsilon) \) for any given positive \( \epsilon \). Finally, let’s see how to use this knowledge to write out a formal proof that \( \lim_{x \to \infty} \frac{x}{x+1} = 1 \).

**Proof.** Let \( \epsilon > 0 \) be given. Choose \( X(\epsilon) = \frac{1}{\epsilon} - 1 \). If \( x > X(\epsilon) = \frac{1}{\epsilon} - 1 \), then \( x + 1 > \frac{1}{\epsilon} \) or \( \frac{1}{x+1} < \epsilon \). Thus for \( x > X(\epsilon) \),

\[
1 - \frac{x}{x+1} = \frac{1}{x+1} < \epsilon.
\]

Since this is true for any \( \epsilon > 0 \), \( \lim_{x \to \infty} \frac{x}{x+1} = 1 \).

**Remark.** The choice of \( X(\epsilon) \) just made is the smallest one possible, since for \( x = \frac{1}{\epsilon} - 1 \), \( 1 - \frac{x}{x+1} = \epsilon \). But any larger choice of \( X(\epsilon) \) would also be valid. For instance, we could just choose \( X(\epsilon) = \frac{1}{\epsilon} \) and it would certainly be true that \( x > X(\epsilon) \) implies \( 1 - \frac{x}{x+1} < \epsilon \).

Now we’ll try something a little more complicated.

**Example 2.** Show that

\[
\lim_{x \to \infty} \left( 1 + \frac{\sin x}{x+1} \right) = 1.
\]
Note that the function \( 1 + \frac{\sin x}{x + 1} \) does not approach 1 from one side, but crosses the horizontal line \( y = 1 \) repeatedly.

![Graph of \( 1 + \frac{\sin x}{x + 1} \) vs. \( x \)](image)

\[
1 + \frac{\sin x}{x + 1} \to 1 \text{ as } x \to \infty
\]

First we figure out what to do. We will want

\[
\left| 1 - \left( 1 + \frac{\sin x}{x + 1} \right) \right| = \frac{|\sin x|}{|x + 1|} < \epsilon.
\]

Here we cannot simply remove the absolute value signs, because the expression inside changes sign repeatedly as \( x \) gets large, and even if we could, we could not solve the resulting inequality for \( x \) anyway. Thus we must modify our strategy a bit. The idea is to replace the object that we must ultimately show to be small (less than \( \epsilon \)) by one that is easier to work with, and which is larger in absolute value than \( \frac{|\sin x|}{x + 1} \) so that if we can show that our new larger object is less than \( \epsilon \), it will follow automatically that \( \frac{|\sin x|}{x + 1} < \epsilon \).

In this case it is simple to do this. We need only note that for any \( x \), \( |\sin x| \leq 1 \). Thus for any \( x > 0 \),

\[
\frac{|\sin x|}{x + 1} = \frac{|\sin x|}{|x + 1|} \leq \frac{1}{x + 1}
\]

and we already know how to deal with this. We can write out a formal proof like this.

**Proof.** Let \( \epsilon > 0 \) be given. Choose \( X(\epsilon) = \frac{1}{\epsilon} - 1 \). If \( x > X(\epsilon) = \frac{1}{\epsilon} - 1 \), then (as before) \( \frac{1}{x + 1} < \epsilon \). Thus for \( x > X(\epsilon) \),

\[
\left| 1 - \left( 1 + \frac{\sin x}{x + 1} \right) \right| = \frac{|\sin x|}{|x + 1|} \leq \frac{1}{x + 1} < \epsilon.
\]

Since this is true for any \( \epsilon > 0 \), \( \lim_{x \to \infty} \left( 1 + \frac{\sin x}{x + 1} \right) = 1 \).
Remark. This second example is more typical of limit proofs. Ideally one would simply like to find \( X(\epsilon) \) so that \( |L - f(x)| < \epsilon \) for \( x > X(\epsilon) \) by solving the inequality \( |L - f(x)| < \epsilon \) for \( x \). In practice this is generally not possible. In that case, the usual strategy is to replace the quantity \( |L - f(x)| \) by some larger quantity which is simple enough that we can “solve for \( x \)” but which is not so much larger that it no longer approaches 0. Then we have \( |L - f(x)| \) trapped. If it is less than something which is less than \( \epsilon \), then it must itself be less than \( \epsilon \). What will vary from proof to proof is the precise way in which we replace \( |L - f(x)| \) by some suitable larger quantity. This will depend on the nature of the function \( f \) and the precise nature of the limit involved (limit at \( \infty \), limit at \(-\infty \), limit at some finite point like \( x = 3 \), etc.) Note that this is also very like what we did in considering the error between upper and lower sums. In the simplest cases we could compute the error directly. In more complicated cases we were reduced to replacing the error by some larger but simpler quantity which we were then able to estimate directly.

Unlike the situation in the previous example where \( X(\epsilon) \) marked the exact largest value of \( x \) for which \( |L - f(x)| = \epsilon \), the value for \( X(\epsilon) \) we have come up with this time is presumably not the smallest possible, at least for most values of \( \epsilon \). For instance, for \( \epsilon = .05 \), we find \( X(.05) = 19 \). But

\[
1 - \left( 1 + \frac{\sin 19}{19 + 1} \right) = \left| \frac{\sin 19}{20} \right| \approx .0075
\]

and according to my calculator, the largest value of \( x \) for which \( \left| \frac{\sin x}{x + 1} \right| = .05 \) is approximately 17.65. So we don’t have the “best” value for \( X(\epsilon) \). But for our purposes we really don’t need the best value, or even any closer approximation to the best value than the one we had, and the choice we have made has the advantage of relative simplicity. In general it is always best to opt for relative simplicity by replacing a more complicated expression by a simpler one (as long as it does give some value for \( X(\epsilon) \)) rather than to strive for the “best” \( X(\epsilon) \).
Example 3. Here is one last example where it is desirable to replace the given expression by a larger but simpler one. We prove

\[
\lim_{x \to \infty} \frac{3x^2 - 1}{x^2 + x + 4} = 3.
\]

Again the first step is to figure out how to proceed by looking at \(\left| \frac{3x^2 - 1}{x^2 + x + 4} - 3 \right|\).

We have

\[
\left| \frac{3x^2 - 1}{x^2 + x + 4} - 3 \right| = \left| \frac{3x^2 - 1 - 3(x^2 + x + 4)}{x^2 + x + 4} \right| = \left| \frac{-3x - 13}{x^2 + x + 4} \right| = \frac{3x + 13}{x^2 + x + 4}
\]

at least for all \(x > 0\). It is this quantity that we want to make less than the tolerance \(\epsilon\) for all large values of \(x\), but solving the inequality \(\frac{3x + 13}{x^2 + x + 4} < \epsilon\) for \(x\) in terms of \(\epsilon\) with the quadratic formula would be an awful mess. So we will replace this rational function by a simpler one that is larger, but not so much larger that it no longer approaches 0. We can make the fraction larger by making the numerator larger and the denominator smaller. In particular, for \(x > 13\), \(3x + 13 < 4x\), and for all \(x > 0\), \(x^2 + x + 4 > x^2\). Thus for \(x > 13\),

\[
\frac{3x + 13}{x^2 + x + 4} < \frac{4x}{x^2} = \frac{4}{x}.
\]

Now we can see how to make the formal argument.

**Proof.** Let \(\epsilon > 0\) be given. Choose \(X(\epsilon) = \frac{4}{\epsilon}\). If \(x > X(\epsilon) = \frac{4}{\epsilon}\), then \(\frac{4}{x} < \epsilon\). Thus for \(x > X(\epsilon)\),

\[
\left| \frac{3x^2 - 1}{x^2 + x + 4} - 3 \right| = \frac{3x + 13}{x^2 + x + 4} < \frac{4}{x} < \epsilon.
\]

Since this is true for any \(\epsilon > 0\), \(\lim_{x \to \infty} \frac{3x^2 - 1}{x^2 + x + 4} = 3\).

**Remark.** Strictly speaking this proof is not quite right. The problem is that one of the inequalities that we used is true only for \(x > 13\). Thus we really should be sure always to pick \(X(\epsilon) > 13\) to be sure that each step in the chain of inequalities is valid. A choice for \(X(\epsilon)\) that takes account of this is to choose \(X(\epsilon)\) to be the greater of \(\frac{4}{\epsilon}\) and 13. In symbols, \(X(\epsilon) = \max \left\{ \frac{4}{\epsilon}, 13 \right\}\). Notice that as in the previous example, this is not the “best” value for \(X(\epsilon)\), but it is good enough.

**EXERCISES.**

1. For each function \(f\) in the list, find a number \(L\) that you believe to be \(\lim_{x \to \infty} f(x)\), and numbers \(X\) such that (i) \(|f(x) - L| < .01\) for all \(x > X\), (ii)
Figure 2-5 Finding $X(\cdot.1)$ by comparison

$$|f(x) - L| < 10^{-6} \text{ for all } x > X, \text{ and (iii) } |f(x) - L| < \epsilon \text{ for all } x > X \text{ (i.e. 3 different } X's).$$

(a) $f(x) = \frac{x}{x - 1}$, (b) $f(x) = \frac{3x}{x - 2}$, (c) $f(x) = \frac{2x}{x^2 + x + 1}$, (d) $f(x) = \frac{x + \sin x}{x - 1}$, (e) $f(x) = \frac{2x^2}{x^2 + x + 1}.

2. Prove that $\lim_{x \to \infty} \frac{2x + 1}{x - 1} = 1$.

3. Prove that $\lim_{x \to \infty} \frac{2x + 1}{x - 1} = 2$.

For the following problems, prove that the limit has the value you think it has. Be careful with absolute value manipulations.

4. $\lim_{x \to \infty} \frac{x + 2\sin x}{x + 1}$

5. $\lim_{x \to \infty} \frac{x^2 + x}{x^3 + 3}$

6. $\lim_{x \to \infty} \frac{x^3 - 2x^2 + 1}{x^3 + x + 101}$

7. $\lim_{x \to \infty} \frac{3x^3 + x\cos(2x)}{x^3 + x^2 + 5}$

8. Use the estimates from #3 of section 1.3 to prove that $\lim_{x \to \infty} \left(\sqrt{x + 1} - \sqrt{x}\right) = 0$.

9. Express the difference as an integral and use the estimates provided by upper and lower sums to prove that $\lim_{x \to \infty} \left(\sqrt{x^2 + 2x - x}\right) = 1$. (Hint: $x = \sqrt{x^2}$.)

10. (a) Using your calculator or otherwise, guess the value of $L = \lim_{x \to \infty} x(\ln(x + 1) - \ln x)$.

(b) Prove that your guess is correct as follows: Express $\ln(x + 1) - \ln x$ as an integral and estimate this quantity using upper and lower sums with one term. (Compare problem 1 of Section 1.3). Use these estimates to find simple functions $f$ and $g$ so that $f(x) < x(\ln(x + 1) - \ln x) < g(x)$, and then, given $\varepsilon$ find $X(\varepsilon)$ that will work for both $|L - f(x)|$ and $|L - g(x)|$.

11. Guess the value of $\lim_{x \to \infty} \left(\frac{1}{x}\right)^{\frac{1}{\ln x}}$ and prove that your guess is correct.

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12. Show, using the definition of limit at infinity, that \( \lim_{x \to \infty} \frac{x (3 + \sin x)}{x + 3} \) does not exist. Be careful to identify explicitly what makes a limit at infinity fail to exist and then to show that this happens.

2.2 Limit at Negative Infinity

The definition of limit as \( x \) approaches infinity is easily modified to cover a limit as \( x \) approaches negative infinity. We need only keep in mind that the picture has now altered from following the graph of \( f \) as we go farther and farther to the right on the number line to following the graph of \( f \) as we go farther and farther to the left. Thus saying that some condition is true “eventually” will now mean that it is true to the left of some reference point \( X \) rather than to the right of \( X \). With this preliminary remark, we have the following definition.

**Definition 17** Let \( f \) be defined for all real numbers \( x \) less than some specified \( x_0 \). We say that \( f(x) \) approaches the **limit** \( L \) as \( x \) approaches negative infinity and write

\[
\lim_{x \to -\infty} f(x) = L
\]

if for each \( \epsilon > 0 \) there is a real number \( X(\epsilon) \) such that

\[
|L - f(x)| < \epsilon
\]

whenever \( x < X(\epsilon) \).

**Example 4.** Show that \( \lim_{x \to -\infty} \frac{x}{x+1} = 1 \). This is very similar to Example 1, except that we must be a little careful to remember than we are now interested in values of \( x \) that are “near to negative infinity.” In particular, the interesting values of \( x \) are now negative, and we will have to be a little more careful in manipulating inequalities.

To see how to choose \( X(\epsilon) \) now, note that

\[
\left| 1 - \frac{x}{x+1} \right| = \left| \frac{1}{x+1} \right| = \frac{1}{|x+1|} = \frac{1}{-(x+1)} = -\frac{1}{x+1}
\]

since \( x + 1 < 0 \). Thus we must now satisfy \( \frac{1}{-(x+1)} < \epsilon \). Since \( -(x+1) > 0 \), this inequality is equivalent to \( \frac{1}{\epsilon} < -(x+1) \) or, multiplying through by \( -1 \), \( \frac{1}{\epsilon} > x + 1 \), or \( -1 - \frac{1}{\epsilon} > x \). Thus we can take \( X(\epsilon) = -1 - \frac{1}{\epsilon} \). The picture now looks like this:

Here is the formal proof..
Proof. Let $\epsilon > 0$ be given. Choose $X(\epsilon) = -1 - \frac{1}{\epsilon}$. If $x < X(\epsilon) = -1 - \frac{1}{\epsilon}$, then $x + 1 < -\frac{1}{\epsilon}$, or, since $-(x + 1) > 0$, $-\frac{1}{x + 1} < \epsilon$. Thus for $x < X(\epsilon)$,

$$|1 - \frac{x}{x + 1}| = \left|\frac{1}{x + 1}\right| = -\frac{1}{x + 1} < \epsilon.$$ 

Since this is true for each $\epsilon > 0$, $\lim_{x \to -\infty} \frac{x}{x + 1} = 1$.

Again, let’s try something a bit more complicated.

**Example 5.** Show $\lim_{x \to -\infty} \frac{3x^3 - 1}{x^3 + x} = 3$. As always we must look at

$$|L - f(x)| = \left|3 - \frac{3x^3 - 1}{x^3 + x}\right| = \left|\frac{3x + 1}{x^3 + x}\right| = \frac{3x + 1}{x^3 + x}$$

for all $x < -1$. (Note, however, that both numerator and denominator are negative here. This will get us into trouble if we’re not careful.) It does not look very promising to try to solve the inequality $\frac{3x + 1}{x^3 + x} < \epsilon$ directly. Instead we will try to replace this quantity by a simpler one with a larger absolute value. We can do that by replacing the denominator by a quantity with a smaller absolute value, or by replacing the numerator by a quantity with a larger absolute value, or both. For the numerator, when $x < -1$,

$$|3x + 1| = -3x - 1 = |3x| - 1 < |3x| = 3|x|.$$ 

For the denominator, when $x < -1$,

$$|x^3 + x| = -x^3 - x = |x^3| + |x| > |x^3|.$$ 

Combining these,

$$\frac{3x + 1}{x^3 + x} = \frac{|3x + 1|}{|x^3 + x|} < \frac{3|x|}{|x^3|} = \frac{3}{|x|^2}.$$
But \( \frac{3}{|x|^2} < \epsilon \) is true when \( |x| > \frac{1}{\sqrt{\epsilon/3}} \), or since we are interested in negative \( x \)'s, when \( x < -\frac{1}{\sqrt{\epsilon/3}} = -\sqrt{\frac{3}{\epsilon}} \). Now we are ready for the formal proof.

**Proof.** Let \( \epsilon > 0 \) be given. Choose \( X(\epsilon) = -\sqrt{\frac{3}{\epsilon}} \). If \( x < X(\epsilon) \), then \( |x|^2 > \frac{3}{\epsilon} \) or \( \frac{3}{|x|^2} < \epsilon \). Also, for such \( x \), \( |3x + 1| < 3|x| \) and \( |x^3 + x| > |x|^3 \). Thus for \( x < X(\epsilon) \),

\[
3 - \frac{3x^3 - 1}{x^3 + x} = \frac{|3x + 1|}{|x^3 + x|} < \frac{3|x|}{|x|^3} = \frac{3}{|x|^2} < \epsilon.
\]

Since this is true for each \( \epsilon > 0 \), \( \lim_{x \to -\infty} \frac{3x^3 - 1}{x^3 + x} = 3 \).

**EXERCISES.** Prove that the limit has the value you think it has. Be careful in manipulating absolute value signs to remember that a “typical” \( x \) is now negative with a large magnitude.

1. \( \lim_{x \to -\infty} \frac{x}{x - 1} \)
2. \( \lim_{x \to -\infty} \frac{x}{2x + 1} \)
3. \( \lim_{x \to -\infty} \frac{4x + 3 \sin x}{x - 3} \)
4. \( \lim_{x \to -\infty} \frac{x^2 + 7 \cos 5x}{x^3 + 3} \)
5. \( \lim_{x \to -\infty} \frac{x^2 + x}{x^3 + x^2} \)
6. \( \lim_{x \to -\infty} \frac{x^3 - 2x^2 + 1}{x^3 + x + 101} \)

### 2.3 Improper Integrals

One important and familiar application of these ideas is to improper integrals. We recall from Chapter 1 that Definition 5 applies to a bounded function defined on a closed, bounded interval. If either the function or the interval is not bounded, Definition 5 cannot be applied. Instead we consider the possibility of defining an integral as a limit of the integrals already defined. In this section we consider unbounded intervals. In a later section we will consider unbounded functions.

**Definition 18** Let \( a \) be a fixed real number, and let \( f \) be a real-valued function that is Riemann integrable on \( [a, b] \) for each real number \( b > a \). We define the **improper integral** \( \int_a^\infty f(x) \, dx \) by

\[
\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx
\]
if this limit exists. In this case we say the improper integral converges. If the limit does not exist, the improper integral diverges. Similarly,

\[ \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \]

if this limit exists. Finally

\[ \int_{-\infty}^{\infty} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{0} f(x) \, dx + \lim_{b \to \infty} \int_{0}^{b} f(x) \, dx \]

provided both individual limits exist. We use the converge-diverge terminology in these cases too.

**Remark:** Note that \( \int_{-\infty}^{\infty} f(x) \, dx \) exists only if the two limits each exist individually. For instance \( \int_{-\infty}^{\infty} x \, dx \) diverges even though \( \int_{a}^{\infty} x \, dx = 0 \) for all positive \( a \). In other words, we don’t allow cancellation between what’s going on for positive \( x \)'s and negative \( x \)'s.

**Examples.** Consider \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \). We know that if \( p \neq 1 \),

\[ \int_{1}^{b} \frac{1}{x^p} \, dx = \left. \frac{x^{1-p}}{1-p} \right|_{1}^{b} = \frac{1}{p-1} (1 - b^{1-p}) \].

If \( p > 1 \), \( b^{1-p} \to 0 \) as \( b \to \infty \), so the integral converges to \( \frac{1}{p-1} \). If \( p < 1 \), then \( b^{1-p} \to \infty \) as \( b \to \infty \). Finally,

\[ \int_{1}^{b} \frac{1}{x} \, dx = \ln b \to \infty \text{ as } b \to \infty. \]

Thus

\[ \int_{a}^{\infty} \frac{1}{x^p} \, dx \begin{cases} \text{converges to } \frac{1}{p-1} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases} \]

**Remark.** We are accustomed to thinking of the integral of a non-negative function as representing the area under the graph of the function. Thus, in particular, for any \( b > 0 \), \( \int_{1}^{b} \frac{1}{x^2} \, dx \) is the area under the graph of \( \frac{1}{x^2} \) from \( x = 1 \) to \( x = b \). As \( b \) increases, the region grows to the right, and so of course its area increases. The calculation above shows that these areas approach 1 as \( b \to \infty \). Thus it is natural to regard 1 as the area of the unbounded region between the \( x \)-axis and the graph of \( \frac{1}{x^2} \) for all \( x \geq 1 \).

Notice that the region bounded by the \( x \)-axis and the graph of the function \( \frac{1}{x^p} \) looks roughly the same for all positive values of \( p \). But the improper integral

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\[ \int_a^\infty \frac{1}{x^p} \, dx \] converges for \( p > 1 \) and diverges for \( p \leq 1 \). Thus an unbounded region of this shape may either have a finite area or fail to have a finite area. The only way to tell which is the case is to do a calculation like the one we have just done.

**Example.** To emphasize that limits of this sort do not always behave in the way our intuition would expect, consider the following situation, sometimes called Gabriel’s Horn. Recall that if a curve \( y = f(x) \) is rotated around the \( x \)-axis, then the resulting solid of revolution has circular cross-sections in the direction perpendicular to the \( x \)-axis with radius \( f(x) \) and area \( \pi (f(x))^2 \). For \( f(x) = \frac{1}{x} \), then, the volume of the solid of revolution generated between \( x = 1 \) and \( x = b \) by revolving this curve around the \( x \)-axis is

\[ \int_1^b \pi \frac{1}{x^2} \, dx = \pi \left( 1 - \frac{1}{b} \right). \]

It thus seems natural to regard \( \pi \int_1^b \frac{1}{x^2} \, dx = \pi \) as the volume of the unbounded “horn” extending from \( x = 1 \) indefinitely to the right.

We will see shortly that the surface area of this horn is also given by an improper integral, but one that diverges. Thus Gabriel’s horn has a finite volume, but infinite surface area.

### 2.3.1 Convergence or Divergence by Comparison

So far we have considered improper integrals where we can find an antiderivative and compute the limit directly (if it exists). Often, however, we cannot find an antiderivative. In this situation we proceed in a way similar to what we have done earlier in the chapter—we try to replace a complicated situation by a simpler one that will give us the information we need.

**Example.** Investigate \( \int_1^\infty \frac{1}{x^4 + \sin x + 1} \, dx \). Here we cannot find an antiderivative and consider the behavior of the integrand near the singularity at \( x = 1 \).
derivative, so we cannot evaluate \( \int_1^b \frac{1}{x^4 + \sin x + 1} \, dx \) explicitly. But we can compare it to \( \int_1^b \frac{1}{x^4} \, dx = -\frac{1}{3} x^{-3} \bigg|_1^b = \frac{1}{3} \left( 1 - \frac{1}{b^3} \right) \). Since \( \frac{1}{x^4 + \sin x + 1} \leq \frac{1}{x^4} \):

\[
\int_1^b \frac{1}{x^4 + \sin x + 1} \, dx \leq \int_1^b \frac{1}{x^4} \, dx = \frac{1}{3} \left( 1 - \frac{1}{b^3} \right).
\]

(This is Theorem 12 from Chapter 1. Think of the areas, as in the diagram. The area under the higher graph is greater than the area under the lower graph.)

We know that as \( b \to \infty \), \( \int_1^b \frac{1}{x^4} \, dx = \frac{1}{3} \left( 1 - \frac{1}{b^3} \right) \to \frac{1}{3} \). Since the area function \( \int_1^b \frac{1}{x^4 + \sin x + 1} \, dx \) is increasing as \( b \) increases, but is always less than \( \int_1^b \frac{1}{x^4} \, dx \) and so always less than \( \frac{1}{3} \), it seems reasonable that \( \int_1^b \frac{1}{x^4 + \sin x + 1} \, dx \) must also approach a limit as \( b \to \infty \) and that this limit must be less than \( \frac{1}{3} \). What we need is the following theorem applied to the function \( f(b) = \int_1^b \frac{1}{x^4 + \sin x + 1} \, dx \).

**Theorem 19** Let \( f \) be a non-decreasing function defined for all \( x \) greater than some \( a \). If there is a number \( M \) such that \( f(x) \leq M \) for all \( x > a \) then \( \lim_{x \to \infty} f(x) \) exists and \( \lim_{x \to \infty} f(x) \leq M \).

Remark. This limit result differs from those we have seen before in that we do not know what number \( f \) is supposed to converge to. To find it is the hard part of the proof. This requires a deep property of the real numbers that we will introduce in Chapter 3. For the moment we just note that as in the very first example in this chapter, once a non-decreasing function gets close (say, within \( \varepsilon \)) to its limit, the non-decreasing property forces to stay at least that close for all larger values of \( x \). It cannot wobble closer to and then farther away.
from the limit as an oscillating function can. Thus if we could just identify the limit, the rest would be easy. We will return to this point and complete this proof in Chapter 3.

Remark. The theorem has this important consequence for improper integrals: if \( f \) is a non-negative function, then \( \int_a^\infty f(x) \, dx \) diverges only if \( F(b) = \int_a^b f(x) \, dx \) is not bounded above as \( b \) increases. (Intuitively, the only way that the improper integral of a non-negative function can diverge is that the area under its graph grows without bound as \( b \to \infty \).) This is so because the theorem states that as soon as \( F(b) \) is bounded above, \( \lim_{b \to \infty} F(b) = \int_a^\infty f(x) \, dx \) must exist.

Example concluded. We know that \( f(b) = \int_1^b \frac{1}{x^4 + \sin x + 1} \, dx \) is an increasing function of \( b \) such that \( f(b) < \frac{1}{3} \) for all \( b \). It then follows from the previous theorem that the improper integral \( \int_1^\infty \frac{1}{x^4 + \sin x + 1} \, dx \) converges, though we do not know the exact value of the limit. We know only that it is less than \( \frac{1}{3} \).

The comparison method of the previous example can be formalized as the following theorem.

**Theorem 20** Let \( f \) and \( g \) be positive Riemann integrable functions defined for all \( x \geq a \), for some \( a \), such that \( f(x) \geq g(x) \) for all \( x \geq a \).

If \( \int_a^\infty f(x) \, dx \) converges, then \( \int_a^\infty g(x) \, dx \) converges.

If \( \int_a^\infty g(x) \, dx \) diverges, then \( \int_a^\infty f(x) \, dx \) diverges.

Remark. Intuitively, the theorem just says that if a larger area is finite, then a smaller one is finite, and conversely, that if a smaller area is infinite, then a larger one is infinite.

Proof. Suppose \( \int_a^\infty f(x) \, dx \) converges and \( \int_a^\infty f(x) \, dx = L \). Then for any \( b > a \),

\[
\int_a^b g(x) \, dx \leq \int_a^b f(x) \, dx \leq \int_a^\infty f(x) \, dx = L.
\]

Since \( G(b) = \int_a^b g(x) \, dx \) is a non-decreasing function of \( b \), it follows from the previous theorem that \( \lim_{b \to \infty} G(b) = \int_a^\infty g(x) \, dx \) exists and that \( \int_a^\infty g(x) \, dx \leq L \).
Conversely, suppose that \( \int_a^\infty g(x) \, dx \) diverges. By the second remark after the previous theorem this means that for any \( M \), there is some \( b \) such that \( \int_a^b g(x) \, dx \geq M \). (That is, the area under the graph of \( g \) gets arbitrarily large if we let \( b \) get big enough.) Since \( g(x) \leq f(x) \) for all \( x \), it is also true that for any \( M \), there is a \( b \) so that \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \geq M \). Thus \( \int_a^\infty f(x) \, dx \) also diverges.

Effective use of this theorem requires having a reasonable library of improper integrals for which convergence or divergence is already known. The integrals \( \int_a^\infty \frac{1}{x^p} \, dx \) are the most important members of such a library. Then the idea is, given an integral whose convergence or divergence is to be determined and for which an antiderivative cannot be found, to search the library for a suitable comparison. One example of this was given before the theorem. Here is another.

**Example.** (Gabriel’s Horn again) The surface area of the region obtained by revolving \( y = f(x) \) around the \( x \)-axis from \( x = 1 \) to \( x = b \) is
\[
2\pi \int_1^b f(x) \sqrt{1 + \left( f'(x) \right)^2} \, dx.
\]
(Consult a calculus book for this formula.) In particular, for \( f(x) = \frac{1}{x} \), the surface area from \( x = 1 \) to \( x = b \) is
\[
2\pi \int_1^b \frac{1}{x} \sqrt{1 + \left( \frac{1}{x^2} \right)^2} \, dx = 2\pi \int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx.
\]
We certainly can’t find an antiderivative here, but we can notice that \( \sqrt{1 + \frac{1}{x^4}} > 1 \) so that
\[
\int_1^b \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx > \int_1^b \frac{1}{x} \, dx.
\]
We already know that \( \int_1^\infty \frac{1}{x} \, dx \) diverges, so the larger improper integral also diverges. Thus Gabriel’s Horn has infinite surface area, despite having finite volume. One might try to create the following paradox from this: since the surface area is infinite, we could never paint it with a finite amount of paint. But since the volume is finite, we could just dump a finite amount of paint into the horn and fill it up, thus painting it. What is wrong with this?

**EXERCISES.**

Decide whether the improper integrals converge or diverge. Evaluate the integral if it converges.

1. \( \int_0^\infty e^{-2x} \, dx \)
2. \( \int_0^1 e^{-2x} \, dx \)
3. \( \int_1^\infty \frac{1}{(2x-1)^{2/3}} \, dx \)
4. \( \int_1^\infty \frac{1}{(x+5)^{3/2}} \, dx \)
5. \( \int_{-\infty}^1 \frac{1}{(2x-3)^2} \, dx \)
6. \( \int_{-\infty}^\infty x \, dx \)
7. \int_{-\infty}^{\infty} \sin \pi x \, dx \\
8. \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \\
9. \int_{0}^{\infty} \frac{x^3}{1 + x^4} \, dx

10. You are sliding down an icy slope on Mt. Baker. By dragging everything you’ve got, you achieve a speed of \(200e^{-t/2}\) ft/sec starting from the time you fell.

(a) Will you ever come to a complete stop according to this model?

(b) When you fell, you were 500 feet from a cliff. Will you go over the edge? If so, when?

11. Same questions as #10 if your speed is \(200e^{-t/3}\) ft/sec.

Use an appropriate comparison to decide whether each of these improper integrals converges or diverges. Justify your conclusion.

12. \int_{1}^{\infty} \frac{2 + \cos x}{x^2 + 5} \, dx \\
13. \int_{2}^{\infty} \frac{2 + \sin x}{\sqrt{x} - 1} \, dx \\
14. \int_{10}^{\infty} \frac{1}{\sqrt{x^4 + x^2 + 1}} \, dx

15. \int_{1}^{\infty} \frac{1}{\sqrt{x} (x + 1)} \, dx \\
16. \int_{0}^{\infty} -x^2 \, dx \\
17. \int_{5}^{\infty} \frac{\ln x}{x^2} \, dx

18. \int_{1}^{\infty} \frac{e^{-x} (\sin x)^2}{x + 1} \, dx \\
19. \int_{2}^{\infty} \frac{x}{\sqrt{x^4 - 1}} \, dx

20. \int_{1}^{\infty} \frac{x}{\sqrt{x^4 + 1}} \, dx (be careful to use comparison correctly)

21. Discuss the Gabriel’s Horn paradox mentioned in the example just above.

**Remark.** It should be realized that the function \(f\) in the definition of improper integral need not be non-negative valued. Thus, for instance, it makes sense to investigate whether

\[\int_{0}^{\infty} \frac{\sin x}{x} \, dx\]

converges or diverges. We will return to this example later after studying infinite series, and see that it can be investigated by associating an infinite series with it. More important, however, is the opposite direction – investigating the convergence of an infinite series by comparing it with the improper integral of an appropriate positive function.

### 2.3.2 One Limit From Another—A Preview

In discussing the convergence of \(\int_{0}^{\infty} e^{-x^2} \, dx\) by comparison, one natural comparison to make is that \(e^{-x^2} \leq e^{-x}\) for all \(x \geq 1\). We know \(\int_{0}^{\infty} e^{-x} \, dx\) converges because we can actually compute \(\int_{1}^{\infty} e^{-x} \, dx = e^{-1}\). Thus \(\int_{1}^{\infty} e^{-x^2} \, dx\) exists by comparison and \(\int_{1}^{\infty} e^{-x^2} \, dx < \int_{1}^{\infty} e^{-x} \, dx\). But what about \(\int_{0}^{\infty} e^{-x^2} \, dx\)? The
inequality $e^{-x^2} \leq e^{-x}$ is false for $0 < x < 1$ since $x^2 < x$ for these values of $x$. Thus we cannot use a simple comparison on $[0, \infty)$. On the other hand, a picture suggests that $\int_0^\infty e^{-x^2} \, dx$ should exist and that it should be the sum of two numbers that we know exist.

$$\int_0^\infty e^{-x^2} \, dx = \int_0^1 e^{-x^2} \, dx + \int_1^\infty e^{-x^2} \, dx.$$ 

Since the left side and the second term on the right side of this equation are really limits, a formal justification of this fact is really the following theorem about limits at infinity.

**Theorem 21** Suppose that $\lim_{x \to \infty} f(x)$ exists and that $g(x) = c + f(x)$ for all $x$ greater than some real number $a$. Then $\lim_{x \to \infty} g(x)$ exists and

$$\lim_{x \to \infty} g(x) = c + \lim_{x \to \infty} f(x).$$

**Remark.** We are supposing that $\lim_{x \to \infty} f(x) = L$ and want to prove that $\lim_{x \to \infty} g(x) = c + L$. This is the first example we have seen of a general limit proof, one where we seek to prove something about each function in a whole class of functions. While the general outline will remain the same—given $\epsilon$ we must come up with a procedure for finding $X(\epsilon)$—since we do not have a particular function in mind, the nature of the procedure will necessarily be less explicit.

The idea of the proof is quite simple. The graph of $g$ is just the graph of $f$ translated vertically by the constant $c$. So for any given $x$, the distance between $g(x)$ and $c + L$ is the same as the distance between $f(x)$ and $L$. (See the diagram below.) In particular, whenever $f(x)$ is close to $L$, $g(x)$ is also close to $c + L$. We just have to write this down in the formal language of limits.
Proof. Let $\epsilon > 0$ be given. Since $\lim_{x \to \infty} f(x) = L$, there is $X(\epsilon)$ so that $|f(x) - L| < \epsilon$ for all $x > X(\epsilon)$. For any such $x$,

$$|g(x) - (c + L)| = |(c + f(x)) - (c + L)| = |c - L| < \epsilon.$$  

Thus our choice of $X(\epsilon)$ for $g$ can be the one that works for $f$. Thus $\lim_{x \to \infty} g(x)$ exists and $\lim_{x \to \infty} g(x) = c + L$.

Example continued. Let’s reconsider the improper integral $\int_1^\infty \frac{1}{x^4 + \sin x + 1} \, dx$ from the previous subsection. We saw there that it converged by comparing it with the convergent improper integral $\int_1^\infty \frac{1}{x^4} \, dx$ but remarked that all we could say about its value was that it was less than $\frac{1}{3} = \int_1^\infty \frac{1}{x^4} \, dx$. From the graph there it appears that the two integrands are really pretty close for all $x \geq 2$ and even closer for $x \geq 3$. (In fact $\frac{1}{3^4} \approx .01234$ while $\frac{1}{3^4 + \sin 3 + 1} \approx .01217$.) Thus it might be reasonable to estimate

$$\int_1^\infty \frac{1}{x^4 + \sin x + 1} \, dx = \int_1^3 \frac{1}{x^4 + \sin x + 1} \, dx + \int_3^\infty \frac{1}{x^4 + \sin x + 1} \, dx$$

$$\approx \int_1^3 \frac{1}{x^4} \, dx + \int_3^\infty \frac{1}{x^4} \, dx$$

where the second integral on the right can be computed exactly, and the first integral can be estimated by the methods of Chapter 1, or by more sophisticated numerical methods. (Simpson’s rule, for instance.)

We find $\int_3^\infty \frac{1}{x^4} \, dx = -\frac{1}{3} x^{-3}|_3^\infty \approx .01234$ while $\int_1^3 \frac{1}{x^4 + \sin x + 1} \, dx \approx .19051$ so

$$\int_1^\infty \frac{1}{x^4 + \sin x + 1} \, dx \approx \int_1^3 \frac{1}{x^4 + \sin x + 1} \, dx + \int_3^\infty \frac{1}{x^4} \, dx \approx .19051 + .01234 \approx .2028.$$  

EXERCISES.

1. Use a method like that above to estimate $\int_1^\infty \frac{1}{x^6 + 1} \, dx$. Is your estimate likely to be an overestimate or an underestimate?

2. Use a method like that above and the discussion in the text above to estimate $\int_0^\infty e^{-x^2} \, dx$. Do this first by considering the intervals $[0, 5]$ and $[5, \infty)$ and then the intervals $[0, 10]$ and $[10, \infty)$. What can you conclude about size of the possible error in your estimate?

3. We could examine the size of the error we made in estimating $\int_1^\infty \frac{1}{x^4 + \sin x + 1} \, dx$ above by finding a function smaller than $\frac{1}{x^4 + 1}$ for $x \geq$
3 and comparing its integral over \([3, \infty)\) with the integral of \(x^{-4}\) over that integral. One candidate is \(\frac{1}{(x+1)^4}\) since \((x+1)^4 > x^4 + 2 \geq x^4 + \sin x + 1\) as we can see by multiplying the left side out, noticing that all the terms are positive and that each is greater than 1 for \(x \geq 3\). But we can do better than this by finding a positive number \(a < 1\) so that \((x+a)^4 > x^4 + 2\) for all \(x \geq 3\). One way to do this is to multiply the left side out to get \((x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4\) and to notice that for \(x \geq 3\) and \(a < 1\) the second term is going to be the second largest (why?). We can be sure this is at least 2 by itself if \(4 \cdot 3^3a = 2\) or \(a = \frac{1}{54}\). (Why?) Compute the integral \(\int_3^\infty \frac{1}{(x + \frac{1}{54})^4} dx\),

the difference between this and \(\int_3^\infty \frac{1}{x^4} dx\) and explain carefully why this shows that the estimate above for \(\int_1^\infty \frac{1}{x^4 + \sin x + 1} dx\) is correct to three decimal places.

### 2.4 Limit of a Function at a Point \(a\) – the Basics

Now we turn to the concept of the limit of a function \(f\) at a real number \(a\). Here the intuitive idea is similar – we speak of the values of \(f(x)\) approaching the limiting value \(L\) as \(x\) approaches \(a\) – but the formalism changes slightly to accommodate the fact that approaching \(a\) is a bit different from approaching \(\infty\). Also we must now avoid the pitfall of thinking that the process has something to do with the number \(f(a)\) if this number exists. (Of course it is often the case that the limit of \(f(x)\) as \(x\) approaches \(a\) is equal to \(f(a)\), but we want to avoid building \(f(a)\) into the definition.)

The most obvious reason for avoiding this is that in some very important examples the number \(f(a)\) clearly does not exist.

**Example.** Recall that the derivative at a point \(a\) of a function \(g\), if it exists, is defined to be the limit of the difference quotients

\[
\frac{g(x) - g(a)}{x - a}
\]

as \(x\) approaches \(a\). (Equivalently, the limit of the quotients \(\frac{g(a + h) - g(a)}{h}\) as \(h\) approaches 0. These are in fact the same object, as one sees by making the substitution \(h = x - a\) and contemplating a diagram of what either of these quotients represent.) Think of this as the limit of \(f(x)\) as \(x\) approaches \(a\), where \(f(x) = \frac{g(x) - g(a)}{x - a}\). Since the quotient assumes the form \(\frac{0}{0}\) for \(x = a\), this value is not in the domain of \(f\).

The formal definition looks like this.
**Definition 22** Let $f$ be a function defined for all values of $x$ near a real number $a$ except possibly for $a$ itself. We say that $f(x)$ approaches the limit $L$ as $x$ approaches $a$ and write

$$\lim_{x \to a} f(x) = L$$

if for each $\epsilon > 0$ there is a $\delta > 0$ such that $|L - f(x)| < \epsilon$ whenever $0 < |x - a| < \delta$.

**Remarks.** 1. Note that here the inequality $|x - a| < \delta$ replaces the inequality $x > X(\epsilon)$. Of course this is because $|x - a|$ is the distance between $x$ and $a$ on the number line, so the definition has the form that $f(x)$ is close to $L$ whenever $x$ is close enough to $a$. To be really parallel to the situation at infinity, we should write $\delta(\epsilon)$ to indicate explicitly that $\delta$ depends upon $\epsilon$. We will write $\delta(\epsilon)$ at first for emphasis, but it is more usual to omit an explicit reference to $\epsilon$ and just write $\delta$.

2. The phrase “defined for all values of $x$ near $a$ except possibly for $a$ itself” could be written more formally as “defined for $0 < |x - a| < \delta_0$ for some $\delta_0 > 0$.”

**Example.** Show $\lim_{x \to 2} (3x - 1) = 5$. As before, since we must come up with a suitable $\delta > 0$ for each positive $\epsilon$, we need to find a procedure for determining $\delta$ in terms of $\epsilon$. In a simple case like this one, we will see that the procedure is immediate from the picture. We want to obtain

$$|5 - (3x - 1)| = |6 - 3x| = 3|2 - x| < \epsilon.$$

On a graph of this line (see the diagram), this is just the vertical distance between the y-values on the line at $x$ and at 2 (that is, between the points $(x, 3x - 1)$ and $(2, 5)$).

The other quantity, $|2 - x|$, is just the horizontal distance between the x-values of the same two points. The fact that the ratio of these two quantities...
is equal to 3 is just the statement that this is a line with slope 3. (Note the unequal scales in the diagram above.) Thus we can give the following simple proof.

**Proof.** Let $\epsilon > 0$ be given. Let $\delta = \epsilon / 3$. If $0 < |x - 2| < \delta$, then

$$|5 - (3x - 1)| = |6 - 3x| = 3|x - x| < 3 \cdot \delta = \epsilon.$$ 

Since this is true for each $\epsilon > 0$, $\lim_{x \to 2} (3x - 1) = 5$.

Now we’ll tackle a slightly more complicated example. As with limits at infinity, it will be convenient to replace $|L - f(x)|$ by a larger quantity that is easier to estimate.

**Example.** Show $\lim_{x \to 2} (x^3 - 3x^2 - x + 5) = -1$. Given $\epsilon$, we must find $\delta(\epsilon)$ so that $|1 - (x^3 - 3x^2 - x + 5)| < \epsilon$ whenever $0 < |x - 2| < \delta(\epsilon)$. Now

$$|1 - (x^3 - 3x^2 - x + 5)| = |x^3 - 3x^2 - x + 6| = |x - 2| |x^2 - x - 3|.$$ 

Furthermore, it is not an accident that $x - 2$ is a factor of $(x^3 - 3x^2 - x + 5) - (-1) = x^3 - 3x^2 - x + 6$. This is equivalent to the fact that this quantity is equal to 0 when $x = 2$, that is, that $-1$ is the value of $x^3 - 3x^2 - x + 5$ when $x = 2$.

If we could find a fixed number $K$ such that $|x^2 - x - 3| \leq K$ whenever $|x - 2| \leq 1$ (that is, for $1 \leq x \leq 3$), then we would have that for $|x - 2| \leq 1$,

$$|1 - (x^3 - 3x^2 - x + 5)| = |x - 2| |x^2 - x - 3| \leq K |x - 2|. \quad (2.1)$$

We could then imitate the easy choice of the previous example by choosing $\delta(\epsilon) = \epsilon / K$ for each $\epsilon \leq K$. To obtain such an estimate, note that for $1 \leq x \leq 3$,

$$|x^2 - x - 3| \leq |x|^2 + |x| + 3 \leq 9 + 3 + 3 = 15. \quad (2.2)$$

Of course this is a pretty crude estimate of the maximum of $|x^2 - x - 3|$ over this interval. In fact, by graphing the parabola (or otherwise) we could determine that the actual maximum on this interval is only 3, achieved at both endpoints. $(x^2 - x - 3 = -3$ at $x = 1$, and $= 3$ at $x = 3$.) However for our purposes the more accurate estimate is no real gain, and in any case the cruder method works just as easily for more complicated functions when the actual maximum might be considerably more difficult to compute.

One last remark: our choice $\delta(\epsilon) = \epsilon / 15$ is valid only when the inequalities in equation (2.2) are valid, that is, when $1 \leq x \leq 3$ or $|x - 2| \leq 1$. So we must define $\delta$ in such a way that it is never greater than 1 in order to be sure that (2.2) is correct. We can now proceed with a formal proof.

**Proof.** Let $\epsilon > 0$ be given. Choose $\delta(\epsilon) = \min\{\epsilon / 15, 1\}$. If $|x - 2| \leq 1$, then $|x^2 - x - 3| \leq |x|^2 + |x| + 3 \leq 15$. Thus when $0 < |2 - x| < \delta \leq \epsilon / 15$, we have that

$$|1 - (x^3 - 3x^2 - x + 5)| = |x - 2| |x^2 - x - 3| \leq 15 |x - 2| < 15 \delta \leq \epsilon.$$
Since we have found a suitable $\delta (\varepsilon)$ for each $\varepsilon$, $\lim_{x \to 2} (x^3 - 3x^2 - x + 5) = -1$.

**Remark.** Graphically, the inequality (2.1) says that for $1 \leq x \leq 3$, the graph of $x^3 - 3x^2 - x + 5$ lies between lines of slope $15$ and $-15$ passing through the point $(2, -1)$. Thus we may replace the given cubic function by the steeper (in this interval) linear ones to determine how close we need $x$ to get to $2$ in order for the function value to be within $\varepsilon$ of $-1$.

In more detail, note that from (2.1) we have for $x > 2$ (and $x \leq 3$) by removing the absolute value signs

$$-1 - (x^3 - 3x^2 - x + 5) \leq 15(x - 2) \text{ or } x^3 - 3x^2 - x + 5 \geq -1 - 15(x - 2)$$

and similarly,

$$1 + (x^3 - 3x^2 - x + 5) \leq 15(x - 2) \text{ or } x^3 - 3x^2 - x + 5 \leq -1 + 15(x - 2).$$

This puts the graph of the cubic between the two lines for $2 < x \leq 3$. Similarly, using $|x - 2| = 2 - x$ for $1 \leq x < 2$ we get for these values of $x$,

$$-1 + 15(x - 2) \leq x^3 - 3x^2 - x + 5 \leq -1 - 15(x - 2).$$

Now we’ll try the same thing with a rational function.

**Example.** Show that $\lim_{x \to 0} \frac{1}{1 + x} = 1$. As usual, we start by computing

$$\left| 1 - \frac{1}{1 + x} \right| = \left| \frac{x}{1 + x} \right| = \left| \frac{1}{1 + x} \right| |x - 0|.$$
Here we have written \( x \) as \( x - 0 \) to emphasize the fact that because we are trying to compute the limit at \( x = 0 \), it is \( |x - 0| \) that measures the distance between \( x \) and the limiting \( x \)-value. Now we see that we can repeat the usual argument provided we can find a constant \( K \) so that \( \frac{1}{1+x} \leq K \) for all \( x \) near 0. Since we must stay away from \( x = -1 \) (why?), we consider \( x \) between \(-.5\) and \(.5\). For such \( x \) the greatest value that \( \frac{1}{1+x} \) has is 2 at \( x = -.5 \). (For instance, you can graph \( \frac{1}{1+x} \) to see that.) Now we know that \( 1 - \frac{1}{1+x} \leq 2 |x| \) for \(-.5 \leq x \leq .5\). Graphically, this says that \( \frac{1}{1+x} \) is trapped between the two lines through \((0, 1)\) with slopes 2 and \(-2\) for \(-.5 \leq x \leq .5\).

Now the formal proof looks like this.

**Proof.** Let \( \epsilon > 0 \) be given, \( \epsilon < 1 \). Choose \( \delta (\epsilon) = \epsilon/2 \). If \( |x| < \delta < .5 \), then

\[
\left| 1 - \frac{1}{1+x} \right| = \left| \frac{x}{1+x} \right| < 2 |x| < 2\delta = \epsilon.
\]

Since we have found a suitable \( \delta (\epsilon) \) for each \( \epsilon < 1 \), \( \lim_{x \to 0} \frac{1}{1+x} = 1 \).

**Remark.** It is not hard to see that with any polynomial or rational function \( f(x) \) some version of this technique will work to prove \( \lim_{x \to a} f(x) = f(a) \). The reason is that the quantity \( f(x) - f(a) \) whose size we are trying to estimate is a polynomial or rational function whose value at \( x = a \) is \( f(a) - f(a) = 0 \),
and this means that we can factor this expression: \( f(x) - f(a) = g(x)(x - a) \) where \( g(x) \) is also a polynomial or rational function \( (g(x) = x^2 - x - 3 \) in the previous example) and then we can estimate \(|g(x)|\) for values of \( x \) near \( x = a \) by the method of the example.

Often we can trap other kinds of functions between a pair of lines, though the method of estimation is likely to be a little different and to depend on the specific function being considered.

**Example.** Show \( \lim_{x \to 2} \sqrt{x} = \sqrt{2} \). We must consider

\[
\left| \sqrt{x} - \sqrt{2} \right|
\]

In order to relate this to \(|x - 2|\), I will multiply and divide by \(|\sqrt{x} + \sqrt{2}|\) to “rationalize the numerator.” We get

\[
\left| \sqrt{x} - \sqrt{2} \right| = \frac{|\sqrt{x} - \sqrt{2}| |\sqrt{x} + \sqrt{2}|}{|\sqrt{x} + \sqrt{2}|} = \frac{|x - 2|}{|\sqrt{x} + \sqrt{2}|}.
\]

This looks more hopeful, since there is now a factor of \(|x - 2|\) in the numerator. Also, it is easy to replace the denominator by something simpler since \(|\sqrt{x} + \sqrt{2}| > \sqrt{2}\) for \( x > 0 \). Thus we can construct the following formal argument.

Let \( \epsilon > 0 \) be given. Choose \( \delta = \min \{ \epsilon \sqrt{2}, 2 \} \). (We need \( \delta < 2 \) to be sure that we consider only positive values for \( x \).) If \( 0 < |x - 2| < \delta \), then

\[
\left| \sqrt{x} - \sqrt{2} \right| = \frac{|x - 2|}{|\sqrt{x} + \sqrt{2}|} < \frac{1}{\sqrt{2}} |x - 2| < \frac{\sqrt{2}}{\sqrt{2}} \delta \leq \epsilon.
\]

Since we have found a suitable \( \delta \) for each \( \epsilon > 0 \), \( \lim_{x \to 2} \sqrt{x} = \sqrt{2} \).

However this method will break down at \( x = 0 \).

**Example.** Consider

\[
\lim_{x \to 0} \sqrt{|x|}.
\]

To start with the usual discussion, it seems apparent that the value of the limit is 0 so that the quantity we want to estimate is \( \left| \sqrt{|x|} - 0 \right| = \sqrt{|x|} \). It is clear from the graph of this function that we cannot hope to trap the graph between two lines intersecting at the origin as we have done before, since the function has a vertical tangent line at the origin.
However we can do something else that is very simple: the inequality $\sqrt{|x|} < \epsilon$ is equivalent to $|x| < \epsilon^2$, that is, our function will be less than $\epsilon$ whenever $|x - 0| = |x|$ is less than $\delta = \epsilon^2$. Thus the formal argument is simply the following:

Let $\epsilon > 0$ be given. Choose $\delta(\epsilon) = \epsilon^2$. For $0 < |x| < \delta = \epsilon^2$, we have (by taking square roots on both sides) that $\sqrt{|x|} < \sqrt{\delta} = \epsilon$. Thus for $0 < |x| < \delta$,

$$\left|\sqrt{|x|} - 0\right| = \sqrt{|x|} < \sqrt{\delta} = \epsilon.$$

Since we have found an appropriate $\delta$ for each positive $\epsilon$, we have shown that $\lim_{x \to 0} \sqrt{|x|} = 0$.

Here is a slightly more difficult situation to analyze, though it turns out in the end that the graph is again trapped between a pair of lines.

**Example.** Show that $\lim_{x \to 0} \cos x = 1$. Here part of the difficulty is in deciding what we are able to use. In order to practice estimating with integrals, we will use the fact that $\frac{d}{dx} \cos x = -\sin x$ and the Fundamental Theorem of Calculus. (An alternative method that avoids calculus will appear in the exercises at the end of section 2.6.) The FTC says

$$\int_0^x \sin t \, dt = -\cos t |^x_0 = 1 - \cos x.$$
Now \(|1 - \cos x|\) is precisely the quantity that we want to estimate, so this looks promising. To estimate the integral, just recall that \(-1 \leq \sin t \leq 1\), or \(|\sin t| \leq 1\), so for \(x > 0\),

\[
0 < \int_0^x \sin t \, dt \leq \int_0^x 1 \, dt = x
\]

and for \(x < 0\),

\[
\int_0^x \sin t \, dt = -\int_x^0 \sin t \, dt \leq \int_x^0 1 \, dt = |x|.
\]

Either way, we have

\[
|1 - \cos x| = \left| \int_0^x \sin t \, dt \right| \leq |x|,
\]

that is, the graph of \(\cos x\) is trapped between lines through \((0, 1)\) with slopes 1 and \(-1\). Using this inequality, the formal proof is easy.

**Proof.** Let \(\epsilon > 0\) be given. Choose \(\delta (\epsilon) = \epsilon\). If \(0 < |x| < \delta\), then

\[
|1 - \cos x| = \left| \int_0^x \sin t \, dt \right| \leq |x| < \delta = \epsilon.
\]

Since we have found a suitable \(\delta (\epsilon)\) for each \(\epsilon > 0\), we have proved that \(\lim_{x \to 0} \cos x = 1\).

**EXERCISES.**

1. Prove that each of the following limits has the value you determine for it.
   (a) \(\lim_{x \to -3} (2x + 1)\)  
   (b) \(\lim_{x \to -1} (-3x - 7)\)  
   (c) \(\lim_{x \to -1} (4x^2 + 3)\)
   (d) \(\lim_{x \to 2} \frac{x}{x - 1}\)  
   (e) \(\lim_{x \to 3} \frac{x^2}{x + 1}\)  
   (f) \(\lim_{x \to 0} x^{1/3}\)
   (g) \(\lim_{x \to 0} \left( \frac{1}{|x|} \right)^{3/\ln|x|}\)

2. Give an example of a function \(f\) that is defined for all real numbers and has a limit at every real number except \(x = 0\). Use the definition of limit to justify your claim that your function does not have a limit at 0. (What needs to be the case if \(f\) does not have a limit at 0?)

3. Let \(f(x) = \frac{x + 1}{x^2 - 1}\). Does \(f\) have a limit at \(x = 1\)? Justify your answer.

4. (a) Prove that \(\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}\). (b) Prove that for any \(c \neq 0\), \(\lim_{x \to c} \frac{1}{x} = \frac{1}{c}\).

5. Prove that for any \(a > 0\), \(\lim_{x \to a} \sqrt{x} = \sqrt{a}\).

6. Prove that \(\lim_{x \to 0} \cos 2x = 1\).
7. Prove that for any real number \( a \), \( \lim_{{x \to a}} \cos x = \cos a \). (Suggestion: use the Fundamental Theorem of Calculus as in equation (2.3) to derive the inequality \( |\cos x - \cos a| \leq |x - a| \).)

8. Prove that \( \lim_{{x \to 0}} \frac{1}{\cos x} = 1 \). Suggestions: Use the inequality \( |1 - \cos x| \leq |x| \). It will help to show that \( \left| \frac{1}{\cos x} \right| \leq 2 \) (that is, \( \cos x > \frac{1}{2} \)) for all \( x \) in some interval about 0 by using the proof of \( \lim_{{x \to 0}} \cos x = 1 \) to find a \( \delta_0 \) such that \( |1 - \cos x| < \frac{1}{2} \) for all \( x \) with \( |x| < \delta_0 \).

9. By expressing \( x - \sin x \) as an integral and using the inequality (2.3) show that for \( x > 0 \), \( 0 < x - \sin x < x^2/2 \).

10. Make the appropriate modifications to the signs of your calculations in \#9 to show that for \( x < 0 \), \( |x - \sin x| < x^2/2 \).

11. Show \( \lim_{{x \to 0}} \frac{\sin x}{x} = 1 \) by using the results of \#9 & \#10 to estimate \( \left| \frac{1 - \sin x}{x} \right| \).

12. Show \( \lim_{{x \to 0}} \frac{\ln (1 + x)}{x} = 1 \). Suggestions: First express \( \ln (1 + x) \) as an integral and estimate it. Then divide through by \( x \). Consider \( x > 0 \) and \( x < 0 \) separately.

### 2.5 New Limits From Old—Part I

As with many other subjects in mathematics, while it is necessary to be able to work directly from the definitions, it is often convenient to derive general consequences of the definitions and then refer to them if possible rather than going back to the beginning every time. (An obvious case of this is the standard list of differentiation rules. We do not in practice compute derivatives directly from the definition except when the function to be differentiated cannot be gotten from our standard list of “calculus functions” by the operations covered by the rules.)

The rules for limits tend to be of two kinds. One type says that if the limit of some function exists (or several limits in some cases) then the limit of some other related function also exists and has the value that you would naturally expect. The other main type of result is a comparison theorem that allows us to establish the limits of awkward functions by comparing them to simpler functions. Our technique of comparing polynomial and rational functions (and some trig functions) to linear functions in the previous section was a simple example of such a comparison.

Establishing theorems of this kind involves a major step forward in sophistication. Up to this point we have been trying to show that limits exist directly...
from the definition. Now for the first time we are going to assume that some limit exists and try to make use of this information to establish the existence of some other limit. Remember that to establish the existence of a limit, we had to come up with a procedure for finding \( \delta (\epsilon) \) that will work for any \( \epsilon \) that is given to us. If we assume the existence of a limit, then we are assuming the existence of such a procedure, though we may not know explicitly what it is. It is this procedure we will need to use in order to construct a new procedure for the limit whose existence we are trying to establish.

We have already considered one simple example of this kind of argument in the context of limits as \( x \to \infty \) in subsection 2.3.1 where we proved Theorem 21. We’ll start here by rephrasing Theorem 21 to apply to limits as \( x \to a \), a real number \( a \).

**Theorem 23** Suppose that \( \lim_{x \to a} f(x) = L \) and that \( c \) is a real number. Let \( g(x) = f(x) + c \). Then \( \lim_{x \to a} g(x) \) exists and \( \lim_{x \to a} g(x) = L + c \).

**Discussion.** The idea of the proof is just the same as before. We must show that the distance between \( g(x) \) and \( L + c \) is small when \( x \) is close to \( a \). But the distance between \( g(x) \) and \( L + c \) is exactly the same as the distance between \( f(x) \) and \( L \) and we are assuming precisely that there is a procedure for choosing \( \delta \) so that \( |f(x) - L| < \epsilon \) whenever \( 0 < |x - a| < \delta \). Since the distances are the same, we can use the \( f \)-procedure for \( g \).

**Proof.** Let \( \epsilon > 0 \) be given. Since \( \lim_{x \to a} f(x) = L \), there is \( \delta > 0 \) so that \( |f(x) - L| < \epsilon \) whenever \( 0 < |x - a| < \delta \). Now

\[
|g(x) - (L + C)| = |(f(x) + c) - (L + c)| = |f(x) - L| . \text{ Thus for } 0 < |x - a| < \delta , \text{ we have}
\]

\[
|g(x) - (L + C)| = |f(x) - L| < \epsilon .
\]

Since we can find a \( \delta \) for each \( \epsilon > 0 \), we have shown that \( \lim_{x \to a} g(x) = L + c \).

**Afterword.** It cannot be emphasized too strongly that while we do have a procedure here for producing \( \delta (\epsilon) \) from \( \epsilon \), it is not a simple formula of the type that we have found up to now. The procedure is: feed the \( \epsilon \) into the “\( f \)-procedure” and let \( \delta (\epsilon) \) (for \( g \)) be the \( \delta \) that comes out of the \( f \)-procedure. Of course for any specific function \( f \) for which we can write an explicit procedure for \( \delta \), this will then become a specific procedure for \( g \), but the point is that we can make this general argument that applies to all functions \( f \) at once in order to establish the existence of a procedure for \( g \) without having to consider the details of specific functions \( f \). This is typical of the more general limit arguments that we will consider in this section.

We have just seen that the limit of the sum of a function and a constant is easy to relate to the limit of the original function. The limit of the sum of two functions is a little more complicated. Suppose, for instance, we want to examine

\[
\lim_{x \to 2} \left( 3x^2 + \frac{6}{x + 1} \right).
\]
If we are just deciding what the answer is, we will probably do this in two separate parts—we would guess that \( \lim_{x \to 2} 3x^2 = 12 \) and that \( \lim_{x \to 2} \frac{6}{x+1} = 2 \) and then add to guess \( \lim_{x \to 2} \left( 3x^2 + \frac{6}{x+1} \right) = 14 \). But how to establish this formally?

Well, we could just subtract 14 from this sum, make a common denominator of \( x+1 \) for everything and work away at the algebra. But we could avoid most of the algebra by justifying the informal procedure that we went through to guess the limit. To do that we need this theorem.

**Theorem 24** Suppose that

\[
\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M.
\]

Then \( \lim_{x \to a} (f(x) + g(x)) \) exists and

\[
\lim_{x \to a} (f(x) + g(x)) = L + M.
\]

**Discussion.** The commonsense version of the theorem is that the difference between \( f(x) + g(x) \) and \( L + M \) is at worst the sum of the differences between \( f(x) \) and \( L \) and between \( g(x) \) and \( M \). Since each of these individual differences approaches 0 as \( x \) approaches \( a \), so does their sum. The formal proof depends on expressing this idea by using the following inequality for real numbers \( a \) and \( b \).

\[ |a + b| \leq |a| + |b|. \]

(This was Proposition 2 in Section 0.2. Refer to that section for a proof and some discussion.) This inequality is actually an equality when \( a \) and \( b \) have the same sign (either positive or negative) and is a strict inequality when \( a \) and \( b \) have opposite signs. (For instance, \( 3 = |2 + (-5)| \leq |2| + |-5| = 7 \).)

Using this inequality we can estimate

\[
|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|.
\]

We have

\[
|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M|.
\]

From this we see that we can make the left side less than any tolerance (that is, \( \epsilon \)) by making each of the two terms on the right less than half that big. Since we have assumed that each of the individual limits exists, we can do that.

**Proof.** Let \( \epsilon > 0 \) be given. Since \( \lim_{x \to a} f(x) = L \), for each real number \( \epsilon_f > 0 \) there is \( \delta_f > 0 \) such that

\[
|f(x) - L| < \epsilon_f \quad \text{whenever} \quad 0 < |x - a| < \delta_f.
\]
Similarly, since \( \lim_{x \to a} g(x) = M \), for each real number \( \epsilon_g > 0 \) there is \( \delta_g > 0 \) such that
\[
|g(x) - L| < \epsilon_g \text{ whenever } 0 < |x - a| < \delta_g.
\]
Here I have used the subscripted symbols \( \epsilon_f, \delta_f, \epsilon_g, \delta_g \) to emphasize the fact that these are really names of variables and have no particular connection with variable names used in the same context for other functions or at other points. (Just as it does not really make sense to ask whether the \( x \) in \( f(x) = x^2 \) is "the same \( x \)" as the \( x \) in \( g(x) = \sin x \). In each place the symbol "\( x \)" simply indicates a hole into which an appropriate number can be inserted.)

In particular, the first displayed line means that there is such a \( \delta_f \) for \( \epsilon_f = \epsilon/2 \) (this is the \( \epsilon \) given at the beginning of the proof) and the second means that there is such a \( \delta_g \) for \( \epsilon_g = \epsilon/2 \). Choose \( \delta \) to be the smaller of \( \delta_f \) and \( \delta_g \). Then for \( 0 < |x - a| < \delta \),
\[
|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|
\leq |f(x) - L| + |g(x) - M|.
\]
Since for every \( \epsilon > 0 \) we have found a number \( \delta > 0 \) such that
\[
|f(x) + g(x) - (L + M)| < \epsilon
\]
whenever \( 0 < |x - a| < \delta \), we have shown that \( \lim_{x \to a} (f(x) + g(x)) = L + M \).

Thus in a formal proof that \( \lim_{x \to 2} \left( 3x^2 + \frac{6}{x+1} \right) = 14 \) we could just establish separately that \( \lim_{x \to 2} 3x^2 = 12 \) and that \( \lim_{x \to 2} \left( \frac{6}{x+1} \right) = 2 \) and then conclude what we want from Theorem 24.

**Example.** Another labor-saving device has to do with multiplying by a constant. Can we conclude that if, say, \( \lim_{x \to 2} f(x) = 4 \), (as is the case with \( f(x) = x^2 \)) then \( \lim_{x \to 2} 3f(x) = 3 \cdot 4 = 12? \) The key to this is relating \( |3f(x) - 12| \) to \( |f(x) - 4| \). This is easy to do just by factoring the 3 out:
\[
|3f(x) - 12| = |3(f(x) - 4)| = 3|f(x) - 4|.
\]
So we could make \( |3f(x) - 12| < \epsilon \) by making \( |f(x) - 4| < \frac{\epsilon}{3} \), for then
\[
|3f(x) - 12| = 3|f(x) - 4| < 3 \frac{\epsilon}{3} = \epsilon.
\]

The formal argument looks like this. Let \( \epsilon > 0 \) be given. Since \( \lim_{x \to 2} f(x) = 4 \), there is a \( \delta > 0 \) so that \( |f(x) - 4| < \frac{\epsilon}{3} \) whenever \( 0 < |x - 2| < \delta \). When this is true,
\[
|3f(x) - 12| = 3|f(x) - 4| < 3 \frac{\epsilon}{3} = \epsilon.
\]
Since for each \( \varepsilon > 0 \) we have found an appropriate \( \delta > 0 \), we have shown that 
\[
\lim_{x \to 2} 3f(x) = 12.
\]

Of course the following more general version is true with a similar proof.

**Theorem 25** Suppose that \( \lim_{x \to a} f(x) = L \) and that \( c \) is any real number. Then 
\( \lim_{x \to a} cf(x) \) exists and 
\[
\lim_{x \to a} cf(x) = cL.
\]

*Proof.* See Exercise 1 at the end of this section.

A more elaborate way to operate on a function \( f \) is to compose it with another function \( g \). This is a very important situation to consider, since so many functions can be regarded as compositions of “building block” functions. For instance, if 
\[
\lim_{x \to a} f(x) = L,
\]
then it is natural to expect that 
\[
\lim_{x \to a} (f(x))^2 = L^2,
\]
\[
\lim_{x \to a} e^{f(x)} = e^L,
\]
\[
\lim_{x \to a} \ln(f(x)) = \ln L
\]
and so forth. This theorem says, among many other things, that all of these statements are true.

**Theorem 26** Suppose that \( \lim_{x \to a} f(x) = L \), that \( g(f(x)) \) is defined for all \( x \) near \( a \), that 
\[
\lim_{u \to L} g(u) = M
\]
and that \( g(L) = M \). Then 
\[
\lim_{x \to a} g(f(x)) = M.
\]

*Remarks.* 1. Note that the limit for \( g \) is as \( x \) approaches \( L \), the limit of \( f \). The idea, as always with composition, is that \( f \) maps all the numbers near \( a \) to numbers near \( L \), and \( g \) in turn maps numbers near \( L \) to numbers near \( M \). So the composition of the two functions maps numbers near \( a \) to numbers near \( M \).

2. Before looking at the proof, be sure that you can identify the function \( g \) and the value \( M \) for each of the three examples preceding the statement of Theorem 26.

*Proof.* Let \( \varepsilon > 0 \) be given. We must find \( \delta \) so that 
\[
|M - g(f(x))| < \varepsilon
\]
whenever \( 0 < |x - a| < \delta \). We will work back from \( \varepsilon \) to \( \delta \) in two stages. First, since 
\[
\lim_{u \to L} g(u) = M,
\]
there is \( \delta^* \) so that 
\[
|M - g(u)| < \varepsilon
\]
whenever \( 0 < |u - L| < \delta^* \). Second, since \( \lim_{x \to a} f(x) = L \), there is \( \delta \) so that 
\[
|f(x) - L| < \delta^* \quad \text{whenever} \quad 0 < |x - a| < \delta.
\]
For any such value of \( x \) it follows from the first statement (plugging \( f(x) \) in for \( u \)) that 
\[
|M - g(f(x))| < \varepsilon.
\]

Summarizing, we have shown that for any \( \varepsilon > 0 \) there is \( \delta > 0 \) so that 
\[
|M - g(f(x))| < \varepsilon
\]
whenever \( 0 < |x - a| < \delta \). Thus 
\[
\lim_{x \to a} g(f(x)) = M.
\]
(The procedure here for “constructing” \( \delta \) is: plug \( \varepsilon \) into the “\( g \)-limit procedure” and get out a \( \delta \)-value \( \delta^* \). Plug \( \delta^* \) (now playing the role of \( \varepsilon \)) into the “\( f \)-limit procedure” and get out a value \( \delta \). This is the value \( \delta(\varepsilon) \) that we need. This is summarized in the diagram above.)
Theorem 27 If \( \lim_{x \to a} f(x) = L \) where \( L \neq 0 \), then \( \lim_{x \to a} \frac{1}{f(x)} \) exists and equals \( \frac{1}{L} \).

Proof. Let \( g(x) = \frac{1}{x} \). We have seen in a previous exercise that for each \( b \neq 0 \), \( \lim_{x \to b} g(x) = \frac{1}{b} \). This result then follows immediately from Theorem 26 provided that we can verify that \( g(f(x)) = \frac{1}{f(x)} \) is defined for all \( x \) near \( a \).

In other words, we must show that if \( \lim_{x \to a} f(x) = L \) where \( L \neq 0 \), then \( f(x) \neq 0 \) for all \( x \) near \( a \). This fact is important enough to be formulated as a separate lemma.

Lemma 28 If \( \lim_{x \to a} f(x) = L \) where \( L \neq 0 \), then \( f(x) \neq 0 \) for all \( x \) near \( a \).

Discussion. The idea behind this statement is very simple. For \( x \) near \( a \), \( f(x) \) is near \( L \), and moreover by restricting the interval around \( a \) that we look at, we can restrict the values of \( f \) to be as close to \( L \) as we please. So all we have to do is to make sure that \( f(x) \) is closer to \( L \) than \( 0 \) is in order to be certain that \( f(x) \) is not \( 0 \). It is probably easier to be a bit more specific: take as our tolerance (the \( \epsilon \) half the distance between \( L \) and \( 0 \)). Then \( f(x) \) will be closer to \( L \) than that within some \( \delta \)-interval around \( a \), and so \( f(x) \) certainly cannot be zero there. See the diagram below, where I have chosen a negative value for \( L \) in order to remind you that when \( L \) is negative, the distance between \( L \) and \( 0 \) is not \( L \) but \( |L| \).

Proof. Let \( \epsilon = \frac{|L|}{2} \). Since \( \lim_{x \to a} f(x) = L \), there is \( \delta \) so that \( |f(x) - L| < \epsilon = \frac{|L|}{2} \) whenever \( 0 < |x - a| < \delta \). For such \( x \) we must have \( |f(x)| > \frac{|L|}{2} > 0 \).

(It may be best to think of this inequality in two cases, depending on whether...
L > 0 or L < 0. If L > 0, then \( f(x) > L - \frac{L}{2} = \frac{L}{2} \). If L < 0, then \( f(x) < L + \frac{|L|}{2} = -\frac{|L|}{2} < 0 \). Draw diagrams on a number line to see what’s going on.)

**Remark.** Notice that this argument has a slightly different form from the preceding ones. We are still using the assumption that some limit exists, but since we are not using it to prove the existence of some other limit we do not have to construct a procedure. We just have to use one instance of the procedure whose existence we are assuming. Which instance we choose is rather arbitrary. \( \epsilon = |L|/3 \) or \( |L|/10 \) or in fact any single \( \epsilon \) less than \( |L| \) would work equally well with appropriate modifications that would be suggested by the diagram.

Here is another important consequence of the same idea—using the fact that if \( \lim_{x \to a} f(x) \) exists, then near \( a \) the values of \( f(x) \) are near the limiting value.

**Theorem 29** Suppose that \( \lim_{x \to a} f(x) = L \) exists. Then there is a \( \delta_B > 0 \) such that \( |f(x)| < |L| + 1 \) for all \( x \) with \( 0 < |x - a| < \delta_B \).

**Discussion.** In other words, the values of the function \( f \) are bounded “near” \( a \). It is not true that \( f \) must be a bounded function. For instance \( \frac{1}{x} \) is not bounded despite having a limit at each non-zero real number \( a \). But the theorem says that we can always find an interval around \( a \) in which \( f \) is bounded. The example \( \frac{1}{x} \) makes it clear that the size of this interval may be different for different values of \( a \). For instance, for this function and \( a = 1 \) we could pick \( \delta = .5 \), but for \( a = .1 \) we would have to pick \( \delta < .1 \).

**Proof.** The idea is to estimate \( |f(x)| \) in terms of \( |L| \) and the distance between \( f(x) \) and \( L \). The latter is controlled by making a specific choice for \( \epsilon \), say \( \epsilon = 1 \). The details are left as an exercise.

The last two limit theorems of this type concern products and quotients.
**Theorem 30** Suppose that
\[
\lim_{x \to a} f(x) = L \text{ and } \lim_{x \to a} g(x) = M.
\]
Then \(\lim_{x \to a} f(x)g(x)\) exists and
\[
\lim_{x \to a} f(x)g(x) = LM.
\]
In addition, if \(M \neq 0\), then \(\lim_{x \to a} \frac{f(x)}{g(x)}\) exists and
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}.
\]

**Discussion.** We would like to relate \(|LM - f(x)g(x)|\) to the two individual differences \(|L - f(x)|\) and \(|M - g(x)|\) by using the inequality \(|a + b| \leq |a| + |b|\), just as we did for sums. Here, however, it is not nearly so clear how to do this. It turns out that a trick is required—we add and subtract the quantity \(Lg(x)\).

If we do that we see
\[
|LM - f(x)g(x)| = |LM - Lg(x) + Lg(x) - f(x)g(x)|
\leq |L||M - g(x)| + |L - f(x)||g(x)|.
\]
Then we will have to choose \(\delta\)'s so that each of these terms is less than \(\epsilon/2\).

**Proof.** Let \(\epsilon > 0\) be given. Since \(\lim_{x \to a} g(x) = M\), there is \(\delta_g\) so that
\[
|M - g(x)| < \frac{\epsilon}{2(|L| + 1)}
\]
whenever \(0 < |x - a| < \delta_g\) and also (by Theorem 29) \(\delta_B\) so that
\[
|g(x)| < |M| + 1
\]
whenever \(0 < |x - a| < \delta_B\).

Also, since \(\lim_{x \to a} f(x) = L\), there is \(\delta_f\) so that
\[
|L - f(x)| < \frac{\epsilon}{2(|M| + 1)}
\]
whenever \(0 < |x - a| < \delta_f\).

Let \(\delta = \min \{\delta_g, \delta_B, \delta_f\}\). Then all the inequalities are valid when \(0 < |x - a| < \delta\). Thus for such \(x, a\)
\[
|LM - f(x)g(x)| = |L(M - g(x)) + (L - f(x))g(x)|
\leq |L||M - g(x)| + |L - f(x)||g(x)|
\leq |L|\frac{\epsilon}{2(|L| + 1)} + \frac{\epsilon}{2(|M| + 1)}(|M| + 1)
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Since we have found an appropriate $\delta$ for each $\epsilon > 0$, we have shown that 
$$\lim_{x \to a} f(x) g(x) = LM.$$ 

**EXERCISES.**

1. Prove Theorem 25. Be sure to cover negative values of $c$ and $c = 0$. A slick way to do all at once is to divide by $|c| + 1$ instead of $|c|$ at an appropriate point in the proof.

2. Use a proof similar to that of Theorem 24 to prove that 
$$\lim_{x \to a} (f(x) - g(x)) = L - M.$$ 

3. Show that the assumption that $g(L) = M$ in Theorem 26 is necessary by finding specific functions $f$ and $g$ with $a = 0$ and $L = f(0)$ for which the other assumptions of Theorem 26 hold, but the conclusion is false. (Hint: Look at the next problem.)

4. Explain carefully why it would be possible to drop the assumption $g(L) = M$ in Theorem 26 provided you add the assumption that there is $\delta_0 > 0$ such that $f(x) \neq L$ whenever $0 < |f(x) - a| < \delta_0$.

5. Prove that if $f$ is a function with $\lim_{x \to \infty} f(x) = 2$, then there is $X > 0$ such that $f(x) > 1$ for all $x > X$. (Suggestions: Use an argument similar to that for Lemma 28 above, but modified to deal with limits at infinity as in Section 2.1 rather than a limit at a finite number $a$.)

6. Complete the proof of Theorem 29.

7. Redo the proof of Theorem 29 using $\epsilon = 1/2$. What different bound for the values of $f$ near $a$ does this give us? Explain how the theorem can be true both with this bound and with the bound originally stated.

8. Prove the second part of Theorem 30 by regarding $f(x) / g(x)$ as a product $f(x) \cdot \frac{1}{g(x)}$, using Theorem 27, and imitating the argument above.

9. Adapt the proof of Theorem 24 to prove that if $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to \infty} g(x) = M$, then $\lim_{x \to \infty} (f(x) + g(x))$ exists and $\lim_{x \to \infty} (f(x) + g(x)) = L + M$.

10. Give an example of functions $f$ and $g$ such that $f + g$ has a limit at $x = 0$, but neither $f$ nor $g$ has a limit at $x = 0$.

11. Give an example of functions $f$ and $g$ such that $fg$ has a limit at $x = 1$, but neither $f$ nor $g$ has a limit at $x = 1$. 

60 LIMITS OF FUNCTIONS
2.6 New Limits from Old—Part II

Example. Now let’s consider a fairly difficult limit, \( \lim_{x \to 0} \frac{\sin x}{x} \). You can easily convince yourself with a graphing calculator that \( \frac{\sin x}{x} \to 1 \) as \( x \to 0 \), but if we try to establish this algebraically we get

\[
\left| \frac{\sin x}{x} - 1 \right| = \left| \frac{\sin x - x}{x} \right|
\]

and there certainly doesn’t seem to be any way to factor \( |x - 0| = |x| \) out of this expression. (Factoring \( x \) out of both terms of the numerator as \( \sin x - x = x (\sin - 1) \) is not allowed!) So we can’t be absolutely certain that \( \frac{\sin x}{x} \) doesn’t approach something close to but not quite equal to 1. To really be sure we will prove that the limit has the value 1 exactly by developing a new general method of computing limits.

In fact there are a number of ways to prove that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). In the exercises for the previous section we used the technique of estimating functions by expressing them as integrals to obtain the inequality

\[
\left| \frac{\sin x - x}{x} \right| < \frac{|x|}{2}
\]

so that, as in many previous examples, we have the quantity we are trying to make small trapped between two lines, here of slope \( \pm \frac{1}{2} \), for which the \( \epsilon - \delta \) calculations are easy.

In this section we will trap \( \frac{\sin x}{x} \) itself, rather than \( \frac{\sin x}{x} - 1 \), between two simpler functions, one of which is not a straight line. That \( \frac{\sin x}{x} \to 1 \) as \( x \to 0 \) will then follow from the fact that each of the simpler functions approaches 1 as \( x \to 0 \), and that being trapped between them means that at any \( x \), \( \frac{\sin x}{x} \) is closer to 1 than at least one of the trapping functions. The picture will look like this:

This idea is formalized in the following theorem.

Theorem 31 (Squeeze Theorem) Suppose functions \( f, g, h \) are all defined for \( x \) near \( a \), that

\[
\lim_{x \to a} g (x) = L \quad \text{and} \quad \lim_{x \to a} h (x) = L \quad \text{(same limit)},
\]

and that for some distance \( \delta_0 \),

\[
g (x) \leq f (x) \leq h (x) \quad \text{for } a < x < a + \delta_0
\]
and either
\[ g(x) \leq f(x) \leq h(x) \quad \text{for } a - \delta_0 < x < a \]

or
\[ h(x) \leq f(x) \leq g(x) \quad \text{for } a - \delta_0 < x < a. \]

Then also
\[ \lim_{x \to a} f(x) = L. \]

**Discussion.** 1. The first case corresponds to the functions \( g \) and \( h \) not crossing at \( x = a \) (as in the diagram above) and the second case to \( g \) and \( h \) crossing at \( x = a \) (as in the diagram below).

2. As in the previous theorems, we are trying to derive the existence of a limit from knowledge of the existence of some related limits. In the Squeeze Theorem we are assuming the existence of two limits and want to use this to show the existence of a third one. Thus once again we will need to use the limit procedures for \( g \) and \( h \), which are known to exist but not explicitly known, in order to “construct” the limit procedure for \( f \). This will again have the following form. Given some \( \epsilon > 0 \), feed it (or something related like \( \epsilon/2 \) or \( \epsilon^2 \) or ...) into the procedures known to exist. The procedures will give you back a number \( \delta' \) with some property (or in this case two numbers \( \delta_1 \) and \( \delta_2 \) each with its own property). From this \( \delta' \) (or \( \delta_1 \) and \( \delta_2 \)) construct the number \( \delta \) needed for the procedure you are trying to construct. In more complicated cases several stages of this sort may be necessary before the desired \( \delta \) finally emerges. In this case we will see it comes immediately.

The proof of the Squeeze Theorem is based on this simple idea: if \( g(x) \leq f(x) \leq h(x) \), then \( f(x) \) is as close to \( L \) as at least one of the numbers \( g(x) \) and \( h(x) \).

**Proof.** There are two cases depending on whether \( g(x) \leq f(x) \leq h(x) \) for \( x < a \) or \( h(x) \leq f(x) \leq g(x) \) for \( x < a \), but the proofs are identical except for trivial differences in wording, so I will assume \( h(x) \leq f(x) \leq g(x) \) for \( x < a \). (This means the functions \( g \) and \( h \) cross at \( a \).)
Let $\epsilon > 0$ be given. By assumption there are two numbers $\delta_1$ and $\delta_2$ so that

$$|g(x) - L| < \epsilon \quad \text{when} \quad 0 < |x - a| < \delta_1,$$

and

$$|h(x) - L| < \epsilon \quad \text{when} \quad 0 < |x - a| < \delta_2.$$

If $\delta$ is the smaller of these two numbers, in symbols $\delta = \min \{\delta_1, \delta_2\}$, then both inequalities are true when $0 < |x - a| < \delta$. That is, when $0 < |x - a| < \delta$, both the functions $g(x)$ and $h(x)$ are trapped between the values $L - \epsilon$ and $L + \epsilon$. (See the diagram above.)

Then using this $\delta$, if $x$ is any fixed real number with $0 < |x - a| < \delta$, $|f(x) - L|$ is less than or equal to at least one of $|g(x) - L|$ or $|h(x) - L|$. (Which one depends on whether $L$ is greater than or less than $f(x)$. If $L \leq f(x)$ as in the diagram, then $|f(x) - L| \leq |h(x) - L|$ if $x > a$ and $|f(x) - L| \leq |g(x) - L|$ if $x < a$. If $L \geq f(x)$, then $|f(x) - L| \leq |g(x) - L|$ if $x > a$ and $|f(x) - L| \leq |h(x) - L|$ if $x < a$.)

Thus, if $0 < |x - a| < \delta$, then $|f(x) - L| \leq \max \{|g(x) - L|, |h(x) - L|\} < \epsilon$. Since for each $\epsilon > 0$ we have found an appropriate $\delta$, $\lim_{x \to a} f(x) = L$ and the proof is complete.

Example. Let’s apply this to show $\lim_{x \to 0} \frac{\sin x}{x} = 1$. We need comparison functions $g$ and $h$. Consider the diagram below where $x$ is the angle at the origin. The area of the smaller triangle (base 1, height $\sin x$) is $\frac{\sin x}{2}$. The area of the larger triangle (base 1, height $\tan x$) is $\frac{\tan x}{2}$. The area of the sector with angle $x$ is $\frac{x}{2\pi} \cdot 1^2 = \frac{x}{2}$. By comparing these areas, we see

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\sin x}{2 \cos x}.$$
Rearranging the left hand inequality gives
\[ \frac{\sin x}{x} < 1 \]
while rearranging the right hand inequality gives
\[ \cos x < \frac{\sin x}{x}. \]
Thus we have our function squeezed between \( g(x) = \cos x \) and \( h(x) = 1 \). The diagram is only valid for \( x > 0 \), but all three of \( \cos x, 1, \) and \( \sin x/x \) are even functions, so any relationship between them that holds for \( x > 0 \) must also hold for \( x < 0 \). (\( \cos x, \frac{\sin x}{x} \) and 1 are the three functions in the diagram just above the statement of the Squeeze Theorem.)

To use the Squeeze Theorem, we must know the limits of the squeezing functions. Here we know \( \lim_{x \to 0} \cos x = 1 \) from the previous section, and \( \lim_{x \to 0} 1 = 1 \) is obvious. Thus by the Squeeze Theorem, \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

**Remark.** Let me emphasize once again the difference between the proofs in this section and our earlier proofs concerning the limits of specific functions. In those specific cases we were usually able to determine the number \( \delta \) explicitly in terms of \( \epsilon \). (In fact, in nearly all the examples so far, we have found \( \delta = \epsilon/K \) for some constant \( K \). But recall \( \lim_{x \to 0} \sqrt{|x|} \) from section 2.4.) In these recent results, though, we do not have an explicit formula for \( \delta \). Instead we have used the non-specific “black box” statement “for every \( \epsilon_f > 0 \) there is a number \( \delta_f > 0 \) such that \( |f(x) - L| < \epsilon_f \) whenever \( 0 < |x - a| < \delta_f \)” as a kind of
function — that is, we have plugged some appropriate number in for \( \epsilon_f (\epsilon/2 \text{ for the theorem about sums, } \epsilon/(1 + |c|) \text{ for the theorem about multiples, } ... \) and used the \( \delta_f \) that the statement gives us to construct the \( \delta \) we need to find.

EXERCISES.

1. Prove this version of the Squeeze Theorem for limits at infinity: If \( g(x) \leq f(x) \leq h(x) \) for all \( x \geq x_0 \) and if \( \lim_{x \to \infty} g(x) = \lim_{x \to \infty} h(x) = L \), then \( \lim_{x \to \infty} f(x) = L \).

2. Use the inequality \( |\sin x| < |x| \) derived above by comparing areas to prove that \( \lim_{x \to 0} \sin x = 0 \).

3. Explain why multiplying and dividing the expression \( \frac{1 - \cos x}{x} \) by \( (1 + \cos x) \), leads to the identity
   \[
   \frac{1 - \cos x}{x} = \frac{\sin^2 x}{x(1 + \cos x)}.
   \]
   Use this identity to prove that \( \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \).

   Remark. The result of exercise 3 is that the derivative of \( \cos x \) at \( x = 0 \) is 0. Similarly \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \) says that the derivative of \( \sin x \) at \( x = 0 \) is 1. If you refer to most calculus books, you will find that these are the only hard parts of a careful proof that \( \frac{d}{dx} \sin x = \cos x \) and \( \frac{d}{dx} \cos x = -\sin x \). (The other ingredients are the addition formulas for \( \sin \) and \( \cos \). You can also try writing this out yourself.)

4. Use the identity from \#3 and Theorems 30 and 27 to prove that \( \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \). Explain why this shows that for small values of \( x \), \( 1 - x^2/2 \) is a good approximation to \( \cos x \). Make a table comparing these two functions for some small values of \( x \). Include enough decimal places for each entry so that the values for \( 1 - x^2/2 \) and \( \cos x \) are not identical.

5. We know that \( \lim_{x \to 0} \left( 1 - \frac{\sin x}{x} \right) = 0 \) and that \( \lim_{x \to 0} (1 - \cos x) = 0 \). Estimate \( \lim_{x \to 0} \frac{1 - \sin x}{1 - \cos x} \) by making a table of values for \( x \) near 0.

2.6.1 A Summary of Limit Theorems

Here, for convenience, is a list of all the theorems from the last two sections.

**Theorem 32** Suppose that \( \lim_{x \to a} f(x) = L \), \( c \) is a real number, and \( \lim_{x \to a} g(x) = M \). Then
1. \( \lim_{x \to a} (f(x) \pm g(x)) = L \pm M, \)

2. \( \lim_{x \to a} cf(x) = cL, \)

3. \( \lim_{x \to a} f(x)g(x) = LM, \)

4. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \) (if \( M \neq 0 \)).

**Theorem 33** Suppose that \( \lim_{x \to a} f(x) = L \), that \( g(f(x)) \) is defined for all \( x \) near \( a \), and that \( \lim_{u \to L} g(u) = M \). Then \( \lim_{x \to a} g(f(x)) = M \).

**Theorem 34** Suppose that \( \lim_{x \to a} f(x) = L \). Then

1. there is \( \delta_B > 0 \) such that \( |f(x)| < |L| + 1 \) whenever \( 0 < |x - a| < \delta_B \),

2. if \( L \neq 0 \), then there is \( \delta_Z > 0 \) such that \( |f(x)| > |L|/2 > 0 \) whenever \( 0 < |x - a| < \delta_Z \).

**Theorem 35** (*Squeeze Theorem*) Suppose functions \( f, g, h \) are all defined for \( x \) near \( a \), that

\[
\lim_{x \to a} g(x) = L \quad \text{and} \quad \lim_{x \to a} h(x) = L \quad \text{(same limit),}
\]

and that for some distance \( \delta_0 \),

\[ g(x) \leq f(x) \leq h(x) \quad \text{for} \quad a < x < a + \delta_0 \]

and either

\[ g(x) \leq f(x) \leq h(x) \quad \text{for} \quad a - \delta_0 < x < a \]

or

\[ h(x) \leq f(x) \leq g(x) \quad \text{for} \quad a - \delta_0 < x < a. \]

Then also

\[
\lim_{x \to a} f(x) = L.
\]
2.7 Improper Integrals Again

We have already considered integrals that are improper because the interval of integration is unbounded. Now we consider briefly integrals that are improper because the integrand is unbounded, like \( \int_0^1 \frac{1}{x} \, dx \). We can again think of this as the area (if it is finite) of an unbounded region in the plane, but this time the region is unbounded vertically rather than horizontally. The idea is essentially the same as before. The improper integral is the limit of ordinary integrals if this limit exists. Here is the definition.

**Definition 36** Let \( b \) be a fixed real number, and let \( f \) be a real-valued function that is Riemann integrable on \([c,b]\) for each real number \( c > a \). We define the improper integral \( \int_a^b f(x) \, dx \) by

\[
\int_a^b f(x) \, dx = \lim_{c \to a} \int_c^b f(x) \, dx
\]

if this limit exists. In this case we say the improper integral **converges**. If the limit does not exist, the improper integral **diverges**. Similarly, if \( f \) is Riemann integrable on \([a,c]\) for each \( c < b \),

\[
\int_a^b f(x) \, dx = \lim_{c \to b} \int_a^c f(x) \, dx
\]

if this limit exists.

**Remark:** Note that for \( \int_{-1}^1 \frac{1}{x} \, dx \) the “bad point” is in the middle, at 0. Just as with \( \int_{-\infty}^{\infty} f(x) \, dx \), we will say in such a case that \( \int_{-1}^1 \frac{1}{x} \, dx \) exists only if the two improper integrals \( \int_{-1}^0 \frac{1}{x} \, dx \) and \( \int_0^1 \frac{1}{x} \, dx \) each exist individually.

**Examples.** Consider \( \int_0^1 \frac{1}{x^p} \, dx \). We know that if \( p \neq 1 \),

\[
\int_a^b \frac{1}{x^p} \, dx = \left. \frac{x^{1-p}}{1-p} \right|_a^b = \frac{1}{1-p} (1 - c^{1-p}).
\]

If \( p < 1 \), \( c^{1-p} \to 0 \) as \( c \to 0 \), so the integral converges to \( \frac{1}{1-p} \). If \( p > 1 \), then \( c^{1-p} \to \infty \) as \( c \to 0 \). Finally,

\[
\int_a^b \frac{1}{x} \, dx = - \ln c \to \infty \text{ as } c \to 0.
\]
Thus
\[
\int_0^1 \frac{1}{x^p} \, dx \begin{cases} 
\text{converges to} \frac{1}{1-p} & \text{if } p < 1, \\
\text{diverges} & \text{if } p \geq 1.
\end{cases}
\]

Notice that the inequalities are exactly opposite to those for \(\int_1^\infty \frac{1}{x^p} \, dx\) except that \(p = 1\) is a divergent case both times.

**Remark.** We are accustomed to thinking of the integral of a non-negative function as representing the area under the graph of the function. Thus, in particular, for any \(c > 0\), \(\int_c^1 \frac{1}{\sqrt{x}} \, dx\) is the area under the graph of \(\frac{1}{\sqrt{x}}\) from \(x = c\) to \(x = 1\). As \(c\) decreases, the region grows to the left and up, and so of course its area increases. The calculation above shows that these areas approach 2 as \(c \to 0\). Thus it is natural to regard 2 as the area of the unbounded region between the \(x\)-axis and the graph of \(\frac{1}{\sqrt{x}}\) for \(0 < x \leq 1\).

Notice that the region bounded by the \(x\)-axis and the graph of the function \(\frac{1}{x^p}\) looks roughly the same for all positive values of \(p\). But the improper integral \(\int_0^1 \frac{1}{x^p} \, dx\) converges for \(p < 1\) and diverges for \(p \geq 1\). Thus an unbounded region of this shape may either have a finite area or fail to have a finite area. The only way to tell which is the case is to do a calculation like the one we have just done.

As with the first kind of improper integral, we evaluate the integral \(\int_c^b f(x) \, dx\) directly with an antiderivative when we can, and then evaluate the limit explicitly. When we cannot find an antiderivative, then we have to proceed by comparison, just as before. The same result holds, for the same reasons.

**Theorem 37** Let \(f\) and \(g\) be positive Riemann integrable functions defined for
a < x ≤ b, such that \( f(x) \geq g(x) \) throughout this interval.

If \( \int_a^b f(x) \, dx \) converges, then \( \int_a^b g(x) \, dx \) converges.

If \( \int_a^b g(x) \, dx \) diverges, then \( \int_a^b f(x) \, dx \) diverges.

**Remark.** Again, the theorem just says that if a larger area is finite, then a smaller one is finite, and conversely, that if a smaller area is infinite, then a larger one is infinite.

**EXERCISES.** State clearly why each integral is improper. Determine whether each converges or diverges and justify your answer. For those that converge, determine the value of the integral if possible.

1. \( \int_0^2 x^{-2/3} \, dx \)
2. \( \int_0^2 (2 - x)^{-2/3} \, dx \)
3. \( \int_e^0 \ln x \, dx \)
4. \( \int_2^4 \frac{1}{\sqrt{x^2 - 4}} \, dx \)
5. \( \int_0^1 \frac{1}{\sqrt{x + \sin x}} \, dx \)
6. \( \int_0^3 (x - 1)^{-2/3} \, dx \).

## 2.8 Continuity at a point \( a \)

**Definition 38** We say that a function \( f \) defined for all points near a point \( a \) is **continuous at \( a \)** if

\[
\lim_{x \to a} f(x) = f(a).
\]

**Remarks.** 1. Note that there are actually several statements about \( f \) contained in the definition above. First, it is asserted that \( \lim_{x \to a} f(x) \) exists. Second, it is part of the original assumption about \( f \) that \( f(a) \) exists. Finally, it is asserted that these two numbers are equal. A function can fail to be continuous because of the failure of any one of these requirements. For instance, each of these functions fails to be continuous at \( x = 1 \):
(i) $f_1(x) = 1$ if $x \geq 1$, $= -1$ if $x < 1$, ($\lim_{x \to 1} f_1(x)$ does not exist)

(ii) $f_2(x) = \frac{x^2 - 1}{x - 1}$, ($f_2(1)$ does not exist)

(iii) $f_3(x) = f_2(x)$ if $x \neq 1$, $= 3$ if $x = 1$. ($\lim_{x \to 1} f_3(x) = 2$, $f_3(1) = 3$)

A more sophisticated example of a function which does not have a limit at a point, despite being bounded near the point is $S(x) = \sin(1/x)$ which does not have a limit at $x = 0$. 
2. The requirement in the definition that \( f \) be defined “for all points near \( a \)” could be stated more formally as: there is \( \delta_0 > 0 \) such that the domain of \( f \) contains the interval \((a - \delta_0, a + \delta_0)\).

3. Of course informally we think of continuity as meaning that we can draw the graph of the function without lifting our pencil off the paper—that is, that the graph is “connected” in some sense. This can be made precise, but that would take us too far afield. In any case, deciding exactly what “connected” means turns out to make this equivalence rather circular.

**Important Remark.** Most of the limit proofs that we have done for specific functions may now be reinterpreted as showing that the function is continuous on its domain. For instance, problem 7 in section 2.4 shows that \( \cos x \) is continuous at each real number \( a \), problem 4 shows that \( \frac{1}{x} \) is continuous at each non-zero number \( a \), and problem 5 shows that \( \sqrt{x} \) is continuous for \( a > 0 \).

Most simple results about continuity follow immediately from the results about limits derived in section 2.5. Here are some examples.

**Theorem 39** Suppose functions \( f \) and \( g \) are continuous at the point \( a \). Then
(a) the function \( f + g \) defined by \((f + g)(x) = f(x) + g(x)\) is continuous at \( a \),
(b) for any real number \( c \), the function \( cf \) defined by \((cf)(x) = cf(x)\) is continuous at \( a \),
(c) the function \( fg \) defined by \((fg)(x) = f(x)g(x)\) is continuous at \( a \),
(d) if \( g(a) \neq 0 \), then the function \( \frac{f}{g} \) defined by \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \) is continuous at \( a \).

**Proof.** Note that this is just the usual notion of what is meant by the addition of functions and multiplication of a function by a real number.

CONTINUITY AT A POINT A
Essentially this is just a special case of Theorem 32 with \( L = f(a) \) and \( M = g(a) \), that is, if \( f \) and \( g \) are each continuous at \( a \), then each has a limit at \( x = a \) and the limits are \( f(a) \) and \( g(a) \) respectively. By Theorem 32, \( f + g \) then also has a limit at \( x = a \) which is the sum of the individual limits:

\[
\lim_{x \to a} (f(x) + g(x)) = f(a) + g(a).
\]

But this is just the statement that \( f + g \) is continuous at \( a \).

The other parts follow similarly from the corresponding parts of Theorem 32.

**Theorem 40** Suppose that \( f \) is continuous at \( a \) and \( g \) is continuous at \( f(a) \). Then the composition \( h(x) = f(g(x)) \) is continuous at \( a \).

**Proof.** This is just Theorem 26 with \( L = f(a) \) and \( M = g(f(a)) \).

**EXERCISES**

1. Sketch the graph of each of the following functions (it’s ok to use your graphing calculator, but be sure it’s not misleading you), and classify it into one of these three categories
   (i) the function is continuous at 0,
   (ii) the function is not continuous at 0, but by giving it the proper value at 0 (what value?) it will become continuous at 0,
   (iii) the function is not continuous at 0, and cannot be made continuous at 0 no matter what value for the function is chosen at 0.

   You do not have to prove anything, but you should be able to defend your choice in terms of the graph of \( f \) and your conclusions about \( \lim_{x \to 0} f(x) \).

   (a) \( f(x) = \frac{\sin x}{x} \)  
   (b) \( f(x) = \frac{\sin x}{\sqrt{|x|}} \)  
   (c) \( f(x) = \frac{\sin x}{x^2} \)  
   (d) \( f(x) = x^{1/3} \)  
   (e) \( f(x) = \sqrt{x + |x|} \)  
   (f) \( f(x) = \frac{x}{|x|} \)  
   (g) \( f(x) = \cos \frac{1}{x} \)

2. Write out the proof of Theorem 40 more carefully, explaining how the requirements of the definition are satisfied and exactly how Theorem 26 is used.
3. LIMITS OF SEQUENCES

3.1 Definition, Basic Properties

We turn now to a discussion of the limit of a sequence of real numbers. Intuitively, a sequence is a list of real numbers, arranged in a certain order. To make precise the notion of being arranged in a certain order, we note that this amounts to saying that there is a first number, a second number, a third number and so forth, that is, that there is an element of the sequence associated with each positive integer. This sounds like a function – one that assigns a number to each positive integer – the n-th element of the sequence is assigned to the positive integer n. This is the form that the formal definition takes.

Definition 41 A sequence of real numbers is a function from a set of the form
\[ \{ n : n \text{ is an integer and } n \geq n_0 \} \]
into the set of real numbers. Here \( n_0 \) is a fixed integer. We write \( \{a_n\}_{n=n_0}^{\infty} \) to denote the sequence which assigns the number \( a_n \) to the integer n.

Remarks. 1. Most often \( n_0 = 1 \), but it is convenient to allow the index set to begin with any integer.
2. Note that by this definition the domain of a sequence is always an infinite set of integers.

Intuitively, a sequence \( \{a_n\}_{n=n_0}^{\infty} \) has a limit \( L \) if the difference \( L - a_n \) approaches zero as \( n \) gets large. As before, we express this formally by demanding that the quantity \( |L - a_n| \) should be less than any preassigned tolerance \( \epsilon \) for all terms \( a_n \) sufficiently far along in the sequence, that is, for all \( a_n \) with \( n \) sufficiently large. The definition looks like this.

Definition 42 A sequence \( \{a_n\}_{n=n_0}^{\infty} \) of real numbers converges to the limit \( L \) if for each \( \epsilon > 0 \) there is an integer \( N_\epsilon \) such that
\[ |L - a_n| < \epsilon \text{ whenever } n > N_\epsilon. \]
We write
\[ \lim_{n \to \infty} a_n = L. \]

Example. Let \( a_n = \frac{n}{n + 1} \). It seems clear that this sequence converges to \( L = 1 \). To see how to demonstrate this, consider how to make \[ 1 - \frac{n}{n + 1} = \]

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\[
\frac{1}{n + 1} < \epsilon. \]
Since all quantities here are positive, we can solve this inequality for \( n \) as \( n > \frac{1}{\epsilon} - 1 \). We would naturally choose this quantity for \( N_\epsilon \) except for the technical difficulty that for nearly all values of \( \epsilon \), the quantity is not an integer. We need to find an integer that is at least as large as \( \frac{1}{\epsilon} - 1 \) and then choose that for \( N_\epsilon \). A convenient notation is

\[ [x] \]
which denotes the ceiling of \( x \). For instance, \([\pi] = 4, [\pi] = -3, [3] = 3\). In general,

\[ [x] - 1 < x \leq [x]. \]
Thus in the present case, \( \frac{1}{\epsilon} - 1 \leq \lfloor \frac{1}{\epsilon} \rfloor \). We can construct a formal argument as follows.

**Proof.** Let \( \epsilon > 0 \) be given. Choose \( N_\epsilon = \lfloor \frac{1}{\epsilon} \rfloor \). Since \( \lfloor \frac{1}{\epsilon} \rfloor > \frac{1}{\epsilon} - 1 \), if \( n > N_\epsilon \), then \( n > \frac{1}{\epsilon} - 1 \) or \( \frac{1}{n} < \epsilon \). Thus if \( n > N_\epsilon \),

\[ \left| 1 - \frac{n}{n + 1} \right| = \frac{1}{n + 1} < \epsilon. \]
Since for each \( \epsilon \) there is \( N_\epsilon \) such that \( \left| 1 - \frac{n}{n + 1} \right| < \epsilon \) whenever \( n > N_\epsilon \), we have proved that \( \lim_{n \to \infty} \frac{n}{n + 1} = 1 \).

How do we check that a sequence does not converge to some number \( L \)? Here is a simple example.

**Example.** Let \( a_n = (-1)^n \) for \( n = 1, 2, 3, \ldots \). Then \( \{a_n\}_{n=1}^\infty \) does not converge to the limit 1 because for, say \( \epsilon = 1 \), no matter how large we try to choose \( N_1 \) we cannot achieve \( |1 - (-1)^n| < 1 \) for all \( n > N_1 \). In fact, \( 1 - (-1)^n = 2 \) for the next (and each) odd integer after \( N_1 \). Thus it is impossible to choose \( N_1 \) with the required property and we must conclude that the limit of \( \{a_n\}_{n=1}^\infty \) is not 1.

We can think of \( \epsilon = 1 \) as a “bad epsilon” in this situation – one for which no \( N_\epsilon \) exists. In fact, a similar argument shows that \( \{(-1)^n\}_{n=1}^\infty \) simply does not have any limit at all, since if \( L \) is any proposed limit, then for \( \epsilon = \max\{|L - 1|, |L + 1|\} \), no \( N_\epsilon \) can be found. Informally, we can think of this as saying that for each real number \( L \) there is a bad epsilon so that no real number can serve as the limit of this sequence.

Limits of sequences clearly behave much like limits of functions defined on an interval of real numbers. In particular, there is a version for sequences of Theorem 32 from section 2.6.
Theorem 43 Suppose that
\[ \lim_{n \to \infty} a_n = L \text{ and } \lim_{n \to \infty} b_n = M. \]

Then
(a) \( \lim_{n \to \infty} (a_n + b_n) = L + M \),
(b) for any real number \( c \), \( \lim_{n \to \infty} ca_n = cL \),
(c) \( \lim_{n \to \infty} a_nb_n = LM \),
and if \( M \neq 0 \),
(d) \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{M} \).

The proofs of these assertions are quite similar to the proofs of the corresponding results given above. To illustrate this, here is the proof of (b).

Proof of (b). Let \( \epsilon > 0 \) be given. Since \( \lim_{n \to \infty} a_n = L \), there is \( N \) so that \( |a_n - L| < \frac{\epsilon}{1 + |c|} \) whenever \( n > N \). For such \( n \),
\[ |ca_n - cL| = |c||a_n - L| < |c|\frac{\epsilon}{1 + |c|} = \frac{|c|}{1 + |c|}\epsilon < \epsilon. \]

Since for any \( \epsilon > 0 \) we have found \( N \) so that \( |ca_n - cL| < \epsilon \) whenever \( n > N \),
\( \lim_{n \to \infty} ca_n = cL \).

There is also a version of Theorem 40 of Section 2.8 on the composition of continuous functions.

Theorem 44 If \( \lim_{n \to \infty} a_n = a \) and if \( f \) is continuous at \( a \), then \( \lim_{n \to \infty} f(a_n) = f(a) \).

Proof. Let \( \epsilon > 0 \) be given. We must find an integer \( N \) so that \( |f(a_n) - f(a)| < \epsilon \) whenever \( n \geq N \). We proceed in two steps, rather as in the proof of Theorem 26 of Section 2.5. First, since \( f \) is continuous at \( a \), there is \( \delta > 0 \) so that \( |f(x) - f(a)| < \epsilon \) whenever \( |x - a| < \delta \). Second, since \( a_n \to a \), there is \( N \) so that \( |a_n - a| < \delta \) whenever \( n \geq N \). Combining these two statements, (that is, using \( a_n \) for \( n \geq N \) in place of \( x \)) we see that \( |f(a_n) - f(a)| < \epsilon \) whenever \( n \geq N \). This is what we needed to prove. Thus the sequence \( \{f(a_n)\}_{n=1}^{\infty} \) converges and has limit \( f(a) \).

Remark. This is an important theorem for the same reason as Theorem 26; we can use a continuous function to build many sequences from one “seed.” For instance, it follows from Theorem 44 that if \( a_n \to a \) then \( a_n^2 \to a^2 \) (\( f(x) = x^2 \)), \( a_n^3 \to a^3 \) (\( f(x) = x^3 \)), \( \sqrt{a_n} \to \sqrt{a} \), \( \cos a_n \to \cos a \), \( \ln a_n \to \ln a \), \( e^{a_n} \to e^a \) and so forth. (What is \( f \) for each of the last four examples?)

EXERCISES.
1. Decide what value each limit has, and prove that this is correct.
(a) \( \lim_{n \to \infty} \frac{n + 1}{n^2 + 5} \),

(b) \( \lim_{n \to \infty} \frac{4n^3 + n \sin n}{n^3 - 5n - 3} \),

(c) \( \lim_{n \to \infty} (\ln(n+1) - \ln n) \) (Use an identity for \( \ln \) and the result of the example at the end of Sec. 1.3.)

(d) \( \lim_{n \to \infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) \)

(e) \( \lim_{n \to \infty} \frac{\sqrt{n + 1}}{2\sqrt{n}} \) (Use Ex. 3 in Section 1.3.)

(f) \( \lim_{n \to \infty} n \sin \left( \frac{1}{n} \right) \),

(g) \( \lim_{n \to \infty} \frac{n^2 - n \cos (n^2) + 6}{2n^2 - 7n - 10} \).

2. Prove that \( \lim_{n \to \infty} \left( \frac{1}{n} \right)^n = 0 \). (Suggestions: take natural logs of what you have to prove. Be careful about the fact that \( \ln x < 0 \) for \( 0 < x < 1 \).)

3. Generalize the result of the previous exercise to prove that \( \lim_{n \to \infty} a^n = 0 \) for each real number \( a \) with \( 0 < a < 1 \).

4. Use the result of the example at the end of Section 1.3 and Theorem 44 to prove that \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e \). You should prove that the sequence \( \{a_n\}_{n=1}^{\infty} \) that you want to apply Theorem 44 to does converge to the limit you assert for it, but you may assume that the function \( f \) you want to use is continuous.

5. Prove part (a) of Theorem 43.

6. Prove part (c) of Theorem 43.

7. Prove that if \( \lim_{n \to \infty} a_n = A \neq 0 \), and if \( \lim_{n \to \infty} a_nb_n = C \), then \( \{b_n\}_{n=1}^{\infty} \) converges.

8. Prove that if \( \lim_{n \to \infty} a_n = L \), then \( \lim_{n \to \infty} |a_n| = |L| \). (Suggestion: Recall Exercise 4 from section 0.2.)

9. Prove the Zipper Theorem: If \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} b_n = L \), then the sequence

\[ a_1, b_1, a_2, b_2, a_3, b_3, \ldots \]

also converges and has limit \( L \). (Note on notation: the combined sequence is \( \{c_n\}_{n=1}^{\infty} \) where \( c_n = \begin{cases} a_k & \text{if } n = 2k-1, \\ b_k & \text{if } n = 2k. \end{cases} \)


### 3.2 Boundedness, Convergence of Monotonic Sequences

Up to this point, we have nearly always proved the existence of a limit involving an explicit formula by identifying a specific candidate for the limit, and then
showing that this candidate does in fact have the properties that the limit of a function (or more recently of a sequence) is supposed to have. (The exceptions were the improper integrals for which we established convergence by comparison rather than by finding an antiderivative. But we were a little vague at the time about justifying this procedure.)

Sometimes, especially when considering infinite series, the identity of the limit is not at all clear. Thus it is very useful to be able to decide whether or not a limit exists without having to know in advance what it is. If one can show that a limit does exist in this way, then one can afterwards try to determine it either exactly, if possible, or at least approximately to some degree of accuracy. This is just what our approach was at the beginning of the course in considering Riemann integrals. We concentrated there on showing that the Riemann integral of a function on a bounded interval exists when the function is monotonic (or changes direction only finitely many times). Here again it will turn out that monotonicity is a very useful property, though the details will be quite different.

**Definition 45** We say that a sequence \( \{a_n\}_{n=n_0}^{\infty} \) is **bounded above** if there is a real number \( M \) such that \( a_n \leq M \) for each \( n \geq n_0 \). Similarly, \( \{a_n\}_{n=n_0}^{\infty} \) is **bounded below** if there is a real number \( m \) such that \( a_n \geq m \) for each \( n \geq n_0 \).

A sequence that is both bounded above and bounded below is said to be **bounded**. Equivalently, \( \{a_n\}_{n=n_0}^{\infty} \) is bounded if there is a positive number \( B \) such that \( |a_n| \leq B \) for each \( n \geq n_0 \).

**Examples.** The sequence \( \{1, 2, 3, \ldots\} \) of positive integers is bounded below (by 1 for instance) but is not bounded above. Similarly, the sequence of negative integers is bounded above, but is not bounded below. The sequence \( \{(-1)^n\}_{n=1}^{\infty} \) is bounded, as is the sequence \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \).

The example of \( \{(-1)^n\}_{n=1}^{\infty} \) shows that a sequence can be bounded without converging to any limit. On the other hand, if a sequence converges, it must be bounded. (This is different from the limit of a function defined on an interval. The difference is roughly that the number of elements in the domain of a sequence is infinite only at infinity.) Formally,

**Theorem 46** If the sequence \( \{a_n\}_{n=n_0}^{\infty} \) converges, then it is a bounded sequence.

**Remark.** Suppose that \( \lim_{n \to \infty} a_n = L \). The idea is that eventually all the terms of the sequence are close to \( L \) and so they can’t get much larger than \( L \) is. However the first few terms of the sequence may not be close to \( L \), so we will have to deal with them in another way. However, any finite set \( \{a_1, a_2, \ldots, a_N\} \) is bounded (by \( \max \{|a_j|: j \leq N\} \)) so it will not be any problem to take care of the first terms separately. Here is the formal proof.
Proof. Since \( \lim_{n \to \infty} a_n = L \), for \( \epsilon = 1 \) there is an integer \( N_1 \) such that 
\[
|L - a_n| < 1 \text{ whenever } n > N_1.
\]
Then \( |a_n| \leq |L| + |a_n - L| < |L| + 1 \) for all \( n > N_1 \). This is a bound for the sequence except for the terms omitted at the beginning of the sequence: \( a_n \) for \( n_0 \leq n \leq N_1 \). Set
\[
M = \max \{|a_{n_0}|, \ldots, |a_{N_1}|\}.
\]
Then for all \( n \geq n_0 \), \( |a_n| \leq B = \max \{M, |L| + 1\} \). Thus \( \{a_n\}_{n=n_0}^\infty \) is a bounded sequence.

Remark. We have seen that we cannot turn this theorem around and conclude that a bounded sequence must converge. However it turns out that for monotonic sequences the truth is simpler: it is true that a bounded monotonic sequence does converge. This is very important in the theory of infinite series (for instance), because it provides a criterion for convergence that does not depend on knowing the identity of the limit in advance. (Note that up to this point we have always identified the limit first, and then verified that it was in fact the limit.) We will see how important this is. First, however, more definitions to make all this precise.

Definition 47. A sequence \( \{a_n\}_{n=n_0}^\infty \) is non-decreasing if \( a_{n+1} \geq a_n \) for each \( n \). It is non-increasing if \( a_{n+1} \leq a_n \) for each \( n \). A sequence which is either non-decreasing or non-increasing is said to be monotonic.

So the assertion is that every bounded, monotonic sequence converges. To prove this, we need to identify the limit in some form. Consider an example.

Example. We have already proved that \( \frac{n}{n+1} \to 1 \). Notice that 1 is an upper bound for this non-decreasing (actually strictly increasing) sequence. It is not the largest element of the sequence – in fact it is not a term in the sequence at all – but it is clearly “right on the edge” of the sequence. Stated slightly more carefully, it is the smallest number that is an upper bound for the sequence, since if \( a \) is any number less than 1, eventually \( a < \frac{n}{n+1} \). (Do you see for a given \( a \) how to identify an integer \( n \) so that this is true? It is worth working it out in order to see how you use the information that \( a < 1 \).)

It turns out that this behavior is typical of non-decreasing, bounded sequences. That is, if we assume that the sequence has some upper bound, then it must have many upper bounds (any number greater than an upper bound is also an upper bound; \( 2, \pi, e, \) and \( 97.4 \) are all upper bounds for \( \left\{ \frac{n}{n+1} \right\} \)).

What is true, and is a deep property of the real numbers, is that such a set of upper bounds always has a smallest member. This is not so obvious as you might think at first. Infinite sets often do not have greatest or smallest elements, even when they are bounded above or below. For the sequence \( \left\{ \frac{n}{n+1} \right\} \), for instance, the sequence itself has no greatest element, though the set of upper bounds for this sequence has the smallest element 1.
As another example of the non-triviality of this assertion, if we were restricted to using only rational numbers, then it would not be true. Consider for example the sequence \( \{a_n\}_{n=1}^{\infty} \) where \( a_n \) is the decimal approximation to \( \pi \) to \( n \) places. Thus \( a_1 = 3.1, a_2 = 3.14, a_3 = 3.141, \ldots \) This is an increasing sequence of rational numbers that is certainly bounded above by a rational number (3.2 for instance), but there is no smallest rational number that is an upper bound for the elements of the sequence. (Of course if we are allowed to use irrational numbers, \( \pi \) itself is the smallest upper bound.)

There is no chance that we can prove that a set of real numbers which is bounded above has a smallest upper bound, or at any rate not without a lengthy detour into the construction of the real numbers from some more primitive set of properties. It is important enough to get special attention, however, so we will adopt it as an axiom. First, however, the official jargon.

**Definition 48** Let \( E \) be a set of real numbers that is bounded above. We say a number \( u \) is the **least upper bound** of \( E \) if

1. for each \( x \in E, x \leq u \), and
2. for each \( v < u \), \( v \) is not an upper bound, that is, there is some \( x \in E \) such that \( x > v \).

Similarly, if \( E \) is bounded below, a number \( l \) is the **greatest lower bound** for \( E \) if

1. for each \( x \in E, x \geq l \), and
2. for each \( m > l \), \( m \) is not a lower bound, that is, there is some \( x \in E \) such that \( x < m \).

In this situation we usually write \( u = \text{lub} E; l = \text{glb} E \).

**Definition 49** The **least upper bound** (or greatest lower bound) of a sequence \( \{a_n\}_{n=1}^{\infty} \) is the least upper bound (or greatest lower bound) of the set \( E = \{a_n : n = 1, 2, \ldots\} \) that is the range of \( \{a_n\}_{n=1}^{\infty} \).

**Axiom.** Each set \( E \) of real numbers that is bounded above has a least upper bound. Each set of real numbers that is bounded below has a greatest lower bound.

**Remark.** Actually we could prove the second of these properties from the first if we were willing to take the time, so it is really necessary to assume only one of them.

**Example.** It is a familiar fact that the average of two real numbers \( a \) and \( b \),

\[ \frac{1}{2}(a + b) = \frac{1}{2}a + \frac{1}{2}b \]

lies in between them — greater than the smaller number but
Figure 3-1

less than the larger one. More generally, for any \( c \) with \( 0 < c < 1 \), \( ca + (1 - c)b \) also lies between \( a \) and \( b \). For instance, if \( a < b \), then

\[
ca + (1 - c)b < cb + (1 - c)b = b.
\]

We can use this simple observation to construct some increasing sequences which are bounded above but whose limit is not so clear.

Let \( a_0 \) be any real number less than 1. (Think of \( a_0 = 0 \) as a first example.) Let \( \{c_n\}_{n=1}^\infty \) be any sequence of numbers between 0 and 1. Define a new sequence \( \{a_n\}_{n=1}^\infty \) recursively as

\[
a_{n+1} = c_{n+1}a_n + (1 - c_{n+1}) \cdot 1, \text{ for } n = 0, 1, 2, \ldots.
\]

This is an increasing sequence whose terms are bounded above by 1. In simple cases we can easily calculate the terms explicitly. For instance, if \( a_0 = 0 \), \( c_n = \frac{1}{2} \) for each \( n \), then

\[
a_1 = \frac{1}{2}, a_2 = \frac{3}{4}, a_3 = \frac{7}{8}, \ldots, a_n = 1 - 2^{-n}, \ldots
\]

and it is clear that the sequence converges to 1. Suppose, however, with \( a_0 = 0 \) we take \( c_n = \sin^2 n \). Then the identity of the limit is much less clear. However it still follows from the next theorem that there is a limit.

**Theorem 50** Every bounded monotonic sequence converges.

**Proof.** We will assume that the sequence \( \{a_n\}_{n=n_0}^\infty \) is non-decreasing. (The proof for non-increasing sequences is symmetric.) By our axiom, the set of numbers \( \{a_n\} \) has a least upper bound \( u \). We will prove that \( a_n \to u \).

Let \( \epsilon > 0 \) be given. If there is no element of the sequence greater than \( u - \epsilon \), then \( u - \epsilon \) would be an upper bound for the sequence. This is impossible, since \( u \) is the least upper bound. Thus there must be an \( N \) such that \( a_N > u - \epsilon \).

Now use the fact that the sequence is non-decreasing. For any \( n \geq N \), \( a_n \geq a_N \). Thus also \( a_n > u - \epsilon \). Since \( u \) is an upper bound for the sequence we have in fact that

\[
u - \epsilon < a_n \leq u
\]

for all \( n \geq N \). Since \( \epsilon \) was arbitrary, we have shown that \( a_n \to u \).

**EXERCISES.**
1. (a) Give an example of a sequence \( \{a_n\}_{n=1}^\infty \) that is bounded above but not bounded below. Is it possible that this sequence converges? Why or why not?

(b) Give an example of a sequence \( \{a_n\}_{n=1}^\infty \) that is bounded above, not bounded below, and not monotonic.

(c) Refine your sequence from (b) so that it has the same properties and in addition, if any finite number of terms at the beginning of the sequence is omitted, the sequence that is left is still not monotonic.

2. Give an example of a sequence \( \{a_n\}_{n=1}^\infty \) with all of the following properties:
   - Lub \( \{a_n\} = 1 \)
   - No individual term \( a_n \) is equal to 1, but the sequence \( \{a_n\}_{n=1}^\infty \) does not converge to 1.

3. Prove Theorem 50 in the case when \( \{a_n\}_{n=n_0}^\infty \) is non-increasing, and the limit is the greatest lower bound of the sequence.

4. Let \( \{a_n\}_{n=1}^\infty \) be the sequence defined by
   \[
   a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+n}.
   \]
   (a) Show that \( \frac{1}{2} \leq a_n < 1 \) for each \( n \).
   (b) Show that \( \{a_n\}_{n=1}^\infty \) is monotonic, and so converges. Can you identify \( \lim_{n \to \infty} a_n \) explicitly?

5. Use the Least Upper Bound Axiom to prove that if a non-decreasing function \( f(x) \), defined for all \( x > x_0 \) is bounded above, that is, there is \( M \) so that \( f(x) \leq M \) for all \( x > 0 \), then \( \lim_{x \to \infty} f(x) \) exists. (You will have to remember how we dealt with limits of functions in Chapter 2! This theorem was stated without proof in our treatment of establishing the convergence of improper integrals by comparison.)

3.3 Recursively Defined Sequences

One important way that sequences are defined is by recursion.

**Definition 51** A recursively defined sequence \( \{x_n\}_{n=0}^\infty \) is a sequence defined by giving its first term \( x_0 \) and a rule \( x_n = f(x_{n-1}) \) for defining each successive term of the sequence in terms of the previous one.

**Examples.**

1. \( x_0 = 0, x_n = x_{n-1} + 1 \). Here \( f \) is the function \( f(x) = x + 1 \). The first few terms are \( x_0 = 0, x_1 = 0 + 1 = 1, x_2 = 1 + 1 = 2, x_3 = 2 + 1 = 3 \). Here it seems apparent that \( x_n = n \) for each \( n \) so that we can convert the recursion formula into an explicit formula for the n-th term \( x_n \).
2. $x_0 = 1, x_n = .9999x_{n-1}$. Here $f$ is the function $f(x) = .9999x$. The first few terms are $x_0 = 1, x_1 = .9999, x_2 = .9999^2 = .99980001, \ldots$. Again we can give a general formula for $x_n, x_n = .9999^n$.

3. $x_0 = 3, x_n = \frac{1}{2} \left( x_{n-1} + \frac{3}{x_{n-1}} \right)$. Here $f$ is the function $f(x) = \frac{1}{2} \left( x + \frac{3}{x} \right)$. The first few terms are $x_0 = 3, x_1 = \frac{1}{2} \left( 3 + \frac{3}{3} \right) = 2, x_2 = \frac{1}{2} \left( 2 + \frac{3}{2} \right) = \frac{7}{4} = 1.75, x_3 = \frac{1}{2} \left( \frac{7}{4} + \frac{12}{7} \right) = \frac{97}{56} \approx 1.73$. Here the situation is more complicated; the terms are getting messy and there is no obvious explicit formula for the $n$-th term.

**Remarks.** 1. Do the sequences just defined have limits? It is clear that the first does not; it is just the sequence of positive integers. The second sequence is not quite so obvious, but a moment’s thought suggests that we can use the ideas of the previous section. Each term of the sequence is .9999 times the previous term, so it is very slightly smaller. This means that the second sequence is decreasing. And all the terms are certainly positive, that is, the sequence is bounded below by zero. Therefore, by the decreasing case of Theorem 50, the sequence does converge. But what is the limit? Theorem 50 gives us only the information that the limit is the greatest lower bound of the sequence. Since 0 is one lower bound, the limit is greater than or equal to 0. But this result will not tell us more than that.

2. The situation for the third sequence is apparently even worse, since it looks so much more complicated. Let’s try to prove that it is bounded below; specifically that $x_n^2 > 3$ for all $n$. This is certainly true for $x_0^2 = 9$ and $x_1^2 = 4$. Now look at the general case:

$$x_n^2 = \frac{1}{4} \left( x_{n-1} + \frac{3}{x_{n-1}} \right)^2 = \frac{1}{4} \left( x_{n-1}^2 + 6 + \frac{9}{x_{n-1}^2} \right)$$

so

$$x_n^2 - 3 = \frac{1}{4} \left( x_{n-1}^2 + 6 + \frac{9}{x_{n-1}^2} \right) - \frac{12}{4} = \frac{1}{4} \left( x_{n-1}^2 - 6 + \frac{9}{x_{n-1}^2} \right) = \frac{1}{4} \left( x_{n-1} - \frac{3}{x_{n-1}} \right)^2 \geq 0.$$ 

Moreover, the square can equal 0 only if the quantity in parentheses equals 0, that is, if $x_{n-1} = \frac{3}{x_{n-1}}$ or (cross-multiplying), $x_{n-1}^2 = 3$. So if $x_{n-1}^2 > 3$ (as is true for $x_0$ and $x_1$) then also $x_n^2 > 3$. It follows that $x_n^2 > 3$ for all $n$. 

LIMITS OF SEQUENCES
We can use this estimate to see that the sequence is also decreasing. We have

\[ x_{n-1} - x_n = x_{n-1} - \frac{1}{2} \left( x_{n-1} + \frac{3}{x_{n-1}} \right) = \frac{1}{2} \left( x_{n-1} - \frac{3}{x_{n-1}} \right). \]

This is positive if \( x_{n-1} > \frac{3}{x_{n-1}} \), that is, if \( x_{n-1}^2 > 3 \). But this is true, so \( x_{n-1} - x_n > 0 \), that is, \( x_{n-1} > x_n \).

Thus again we have a decreasing sequence that is bounded below, so it does converge to some limit. But what is the limit? We can use the next theorem to find out.

**Theorem 52** If the recursively defined sequence \( x_n = f(x_{n-1}) \) converges to some real number \( z \), and if \( f \) is continuous at \( z \), then \( z \) is a fixed point of \( f \), that is, \( z \) is a solution of the equation \( x = f(x) \).

**Remarks.** 1. Theorem 52 is in two respects like the theorem in calculus that if a differentiable function \( f \) has a local maximum or local minimum, then it must occur at a critical point (a point where \( f'(x) = 0 \)). In the first place it does not say whether a recursively defined sequence has a limit or not, just as the calculus theorem does not say whether an arbitrary function has any local extrema. But on the other hand, if you know that there is a limit (as we do for the second and third examples above), then you know that only the fixed points are possibilities. In calculus, only the critical points are possibilities, and in most cases there are not many of those, so it is easy to pick out the proper one. Here too, we can often find the fixed points explicitly (or at least numerically up to any desired accuracy) and then we only have to choose the correct one.

2. Graphically, the fixed points of \( f \) are just the values of \( x \) where the graph of \( f \) touches or crosses the line \( y = x \). So drawing the graph of \( f \) is an effective way to get an idea about possible limits of the recursively defined sequence.

3. To apply the theorem to the second example where \( f(x) = .9999x \), a fixed point would have the property that \(.9999x = x \). The only solution to this linear equation is \( x = 0 \). Since \( f \) has only one fixed point and the sequence \(.9999^n \) must converge to some fixed point, it must be the case that \(.9999^n \to 0 \).

4. For the third example, \( f(x) = \frac{1}{2} \left( x + \frac{3}{x} \right) \). Now \( \frac{1}{2} \left( x + \frac{3}{x} \right) = x \) if \( \frac{1}{2} \cdot \frac{3}{x} = \frac{1}{2} x \) or \( \frac{3}{x} = x \) or \( x^2 = 3 \). Thus \( f \) has two fixed points, \( \sqrt{3} \) and \( -\sqrt{3} \).

Since all the terms in our sequence are positive, it must converge to \( \sqrt{3} \). These two fixed points are the two points where the graph of \( f \) crosses \( y = x \).
Here is the proof of Theorem 52.

Proof. Suppose that \( x_n = f(x_{n-1}) \), that \( x_n \to z \) as \( n \to \infty \), and that \( f \) is continuous at \( z \). Note first that if \( x_n \to z \), then also \( x_{n-1} \to z \) since we have just changed the numbering in the sequence. (More formally, given \( \epsilon > 0 \), if \( |x_n - z| < \epsilon \) for all \( n \geq N \), then \( |x_{n-1} - z| < \epsilon \) whenever \( n - 1 \geq N - 1 \).) Thus by Theorem 44, \( f(x_{n-1}) \to f(z) \). But by definition, \( f(x_{n-1}) = x_n \to z \). Since the sequence \( \{f(x_{n-1})\} \) converges to both \( f(z) \) and \( z \), these two numbers must be the same, that is, \( f(z) = z \) as we were trying to show. (I am using here the “fact” that a sequence can converge to at most one limit. While this seems obvious, strictly speaking it requires proof. Can you give one?)

Remark. So the limit of a recursively defined sequence must be a fixed point of the function \( f \). But it does not work the other way around; just because \( f \) has a fixed point, the sequence does not have to converge to it (or to anything). For instance, consider the recursive sequence \( x_0 = 1, x_n = 2x_{n-1} \). Here \( f(x) = 2x \) which, just like the second example above, has the fixed point \( x = 0 \). But our sequence is \( x_0 = 1, x_1 = 2, x_2 = 2 \times 2 = 4, x_3 = 4 \times 2 = 8 \) and in general, \( x_n = 2^n \). This sequence does not converge. In fact, you can check that for any \( x_0 \neq 0 \), the sequence defined recursively by \( x_n = 2x_{n-1} \) does not converge.

Example. Recursive sequences of the general form \( x_n = ax_{n-1}(1 - x_{n-1}) \) occur in mathematical biology as population models. Here \( x_n \) is the population of some species in year \( n \). Let’s look at the specific case \( x_n = 2x_{n-1}(1 - x_{n-1}) \). Here \( f(x) = 2x(1 - x) \). To see what might happen, we look for fixed points of \( f \). If \( x = 2x - 2x^2 \), then \( 0 = x - 2x^2 = x(1 - 2x) \) so there are two fixed points \( x = 0 \) and \( x = \frac{1}{2} \). (The first one has the interpretation that if there are no animals initially, there will never be any.) What happens if \( x_0 = \frac{1}{4} \)? (Think of the populations as having been scaled in some way; perhaps the units are millions of animals.) Just iterating gives \( x_1 = \frac{3}{8}, x_2 = \frac{15}{32} \). It looks like the
sequence may be increasing and approaching the fixed point $\frac{1}{2}$, that is, the population is stabilizing rather than dying out (approaching 0) or exploding. First we note that $f$ has a maximum value of $\frac{1}{2}$ which occurs only for $x = \frac{1}{2}$. (You don’t need calculus for that; the graph of $f$ is a parabola opening down with vertex half way between its zeros at $x = 0$ and $x = 1$, that is at $x = \frac{1}{2}$.)

So the sequence is bounded above by $\frac{1}{2}$ and in fact if $x_0 \neq \frac{1}{2}$, then $x_n < \frac{1}{2}$ for all $n$. To see that the sequence is increasing,

\[
x_n - x_{n-1} = 2x_{n-1} - 2x^2_{n-1} - x_{n-1} = x_{n-1} - 2x^2_{n-1} = x_{n-1} (1 - 2x_{n-1}) > 0
\]

since both terms in the last product are positive. The sequence is increasing and bounded above so must converge to a fixed point, and it is clear that it converges to $\frac{1}{2}$ rather than 0.

Moreover, essentially the same argument shows that for any initial $x_0$ in the range $0 < x_0 \leq 1$, the sequence will converge to the same limit. First, it is the case that $x_1 < \frac{1}{2}$ (unless $x_0 = \frac{1}{2}$ and the sequence is constant), and then the same argument as before shows that the sequence is increasing from this point on.

The population interpretation would be that this is a very stable situation—starting from any nonzero population, the population will approach the limit $\frac{1}{2}$. 

RECURSIVELY DEFINED SEQUENCES
EXERCISES.

1. Show that the sequence \( x_0 = \sqrt{6}, x_n = \sqrt{6 + x_{n-1}} \) is increasing and bounded above by 6. What is its limit? (What is \( f \) here?)

2. Show that the sequence \( x_0 = 2, x_n = \frac{1}{2} \left( x_{n-1} + \frac{2}{x_{n-1}} \right) \) has the property \( x_n^2 > 2 \) for all \( n \) (so it is bounded below), and is decreasing. What is its limit? How do you know? Use your calculator to compute the first 5 terms in this series. How close have you come to the limit?

3. When I was young, in that prehistoric time before calculators, I was taught the following recursive scheme for approximating the square root of a positive number \( a \). Start with \( x_0 = a \). At each stage, the next term, \( x_n \), is the average of \( x_{n-1} \) and \( \frac{a}{x_{n-1}} \). Prove that this recursively defined sequence does converge to \( \sqrt{a} \).

4. Newton’s method is a method for approximating a root of a differentiable function \( g \). The successive approximations are the terms of a recursively defined sequence where \( x_n = x_{n-1} - \frac{g(x_{n-1})}{g'(x_{n-1})} \). Prove that if this sequence converges to a limit \( z \), then \( g(z) = 0 \). (What is the function \( f \) here?)

5. Consider recursive sequences \( x_n = \frac{3}{2} x_{n-1} (1 - x_{n-1}) \). What are possible limits now? Prove that the sequence converges for \( x_0 = \frac{1}{4} \).

6. Prove that the recursive sequence of \#5 converges for any \( x \) with \( 0 \leq x_0 \leq 1 \).
4. INFINITE SERIES OF REAL NUMBERS

4.1 Definitions; Basic Properties

What do we mean by writing what looks like the sum of an infinite collection of numbers? For instance, consider

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \]

or

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]

It turns out that we can make sense of an expression like this by translating it into a related statement about sequences. Given any sequence \( \{a_k\}_{k=1}^{\infty} \) of real numbers, we can define an associated sequence of partial sums:

\[ S_1 = a_1, \]
\[ S_2 = a_1 + a_2, \]
\[ S_3 = a_1 + a_2 + a_3, \]

and in general

\[ S_n = a_1 + a_2 + \ldots + a_n. \]

Notice that these are perfectly ordinary sums. For the first example above we have \( S_1 = 1, S_2 = 1 + \frac{1}{2} = \frac{3}{2}, S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \) and so forth.

**Definition 53** Let \( \{a_k\}_{k=1}^{\infty} \) be a sequence of real numbers. We say that the infinite series

\[ \sum_{k=1}^{\infty} a_k \]

**converges** if the associated sequence \( \{S_n\}_{n=1}^{\infty} \) of partial sums converges to a limit, \( S \). (Otherwise the series **diverges**.) If the series converges, we call \( S \) the **sum** of the series, and denote it by \( \sum_{k=1}^{\infty} a_k \).

**Remarks.** 1. Thus, symbolically,

\[ \sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k \]
if this limit exists.

2. Note that \( \sum_{k=1}^{\infty} a_k \) is not really a sum. It is a limit of sums, just as an integral is a limit of sums.

3. Although I have written the index set in the definition as beginning with \( k = 1 \), it is often convenient to begin from some other integer. The most common alternative is 0 as in the next example.

**Examples.** 1. If \( a_k = 2^{-k} \) for \( k = 0, 1, 2, \ldots \), then one can easily calculate that \( S_n = 2 - 2^{-n} \). (The first three of these partial sums are just above the Definition, but are numbered \( S_1, S_2, S_3 \) instead of \( S_0, S_1, S_2 \).)

Thus \( 2 = \lim_{n \to \infty} S_n = \sum_{k=0}^{\infty} 2^{-k} \).

2. More generally, let \( r \) be any real number except 1. Consider the finite geometric series

\[
S_n = 1 + r + r^2 + \ldots + r^n.
\]

Note that the defining characteristic here is that there is a common ratio, \( r \), between successive terms. One calculates that \( S_n - rS_n = 1 - r^{n+1} \) (since all other terms cancel) or

\[
S_n = \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r} + \frac{r^{n+1}}{1 - r}.
\]

What happens as \( n \to \infty \)? Well, we know that \( |r^{n+1}| \) will get larger as \( n \) increases if \( |r| > 1 \) and will approach zero if \( |r| < 1 \). In the latter case, \( S_n \to \frac{1}{1-r} \) as \( n \to \infty \). Thus we have

\[
\sum_{k=0}^{\infty} r^k = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1, \\ \text{does not exist} & \text{if } |r| \geq 1. \end{cases}
\]

More generally still, we can adapt to the case of a first term of \( a \) rather than 1 and common ratio \( r \) just by multiplying through by \( a \): if \( |r| < 1 \),

\[
\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.
\]

3. Here is a specific case of a geometric series that you will be familiar with from a long time ago.

What is the meaning of the decimal .33333...? Well, if you just write it out digit by digit, this is

\[
\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots,
\]

that is, it is a geometric series with \( a = \frac{3}{10} \) and \( r = \frac{1}{10} \). Inserting that into the expression above gives

\[
.33333... = \frac{1}{3}
\]

which is what we have been told all along.
4. It is time for an example or two of a series which diverges. Of course \( \sum_{k=1}^{\infty} a_k \) with \( a_k = 1 \) for each \( k \) diverges since then \( S_n = n \) for each \( n \). A more interesting example is the harmonic series:

\[
\sum_{k=1}^{\infty} \frac{1}{k}.
\]

This series does not grow very fast. By my calculator, \( S_{10} = 2.92897 \), \( S_{100} = 5.18738 \), \( S_{200} = 5.87803 \), \( S_{300} = 6.28266 \). Nevertheless, if we are sufficiently patient, \( S_n \) becomes arbitrarily large. (We will see in due course how to estimate which \( n \) to take to give \( S_n \) a prescribed value.) We can see why if we write down the first few terms of the series and group them like this:

\[
1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \left( \frac{1}{9} + \ldots + \frac{1}{16} \right) + \ldots
\]

The point is that each of the two terms in the first set of parentheses is at least \( \frac{1}{4} \), so their sum is at least \( 2 \cdot \frac{1}{4} = \frac{1}{2} \). Similarly, each of the four terms in the next group is at least \( \frac{1}{8} \), so the sum is at least \( 4 \cdot \frac{1}{8} = \frac{1}{2} \). We can continue grouping in this way, and for the same reason, the sum of the \( 2^{n+1} \) terms in the \( n \)-th group is at least \( \frac{1}{2} \). Thus \( S_4 \geq 2, S_8 \geq 2.5, S_{16} \geq 3 \), and in general \( S_{2^n} \geq 1 + \frac{n}{2} \). From this it is clear that the sequence \( \{S_n\} \) is not bounded above and therefore does not converge. One might note also that the sum of each grouped set of terms is certainly less than 1. This leads to the upper bound \( S_{2^n} \leq 1 + n \). These two estimates together suggest that \( S_n \) grows at a rate similar to \( \log_2 n \). We will see an “explanation” of this soon.

**EXERCISES.**

1. Write the repeating decimal \(.9999\ldots\) as a geometric series (what is \( a \)? what is \( r \)?) and use the formula to find the sum. This is the “real explanation” for what you might have been shown with an unconvincing trick in high school.

   Compute the sum of each of these infinite series, if the series converges.

   2. \( \sum_{k=0}^{\infty} \frac{3}{4^k} \)  
   3. \( \sum_{k=1}^{\infty} \frac{3}{4^k} \)  
   4. \( \sum_{k=0}^{\infty} \frac{1}{2^{3k}} \)  
   5. \( \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k \)  
   6. \( \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^{-k} \)  
   7. \( \sum_{k=0}^{\infty} \frac{1}{(2 + \pi)^{2k}} \)

7. Give a specific example of an infinite series with limit 5.

8. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of real numbers that converges to a limit \( L \). What is the \( n \)-th partial sum of the infinite series \( \sum_{k=1}^{\infty} (a_k - a_{k+1}) \)? Guess the limit of this infinite series and prove carefully from the definition that the series converges to this limit.

9. Geometric series can be used to investigate Zeno’s Paradox, which runs as follows: Suppose the hare gives the tortoise a head start and then begins to...
run after him. He can never catch up no matter how much faster he is, as we see from thinking about the sequence of events. After some amount of time, the hare will reach the point from which the tortoise started, but of course the tortoise has moved on. A bit later the hare will reach that point the tortoise had reached at the previous stage, but again the tortoise will have moved on. Clearly we can repeat this scenario as many times as we please and the hare will always arrive at a point from which the tortoise has moved on. Therefore the hare will never catch the tortoise.

To make this more precise, suppose that the tortoise moves at 1 meter per second and the hare at 2 meters per second. The hare gives the tortoise a head start of 16 meters, and they start running at the same time.

(a) Compute the time at which the hare reaches the tortoise’s starting point and the tortoise’s position at that time.

(b) Make a table of the first 4 steps of the process described in the paradox with columns for the amount of time used in that step, the total elapsed time, the distance the hare has moved in that step, the hare’s total distance, the distance the tortoise has moved in that step, and the tortoise’s total distance.

(c) You should see three geometric series emerging in the table. Find the sum of each series.

(d) Explain what the result of (c) tells you about the hare catching the tortoise. (When and where?) How does this compare with the “commonsense” calculation of when the hare will catch the tortoise directly from the information at the beginning of this problem.

(e) Discuss what is wrong with Zeno’s paradox.

10. To calculate the real effect of a 1 billion dollar tax cut, economists assume that recipients will spend most of their extra income, say 80% or $800,000,000, and save the rest. This 80% then becomes extra income for other people, who will also spend 80% or $640,000,000 and save the rest. Imagining that this process continues forever, calculate the total extra income generated by the tax cut. (This is called the multiplier effect in economics.)

11. Suppose in the situation of the previous problem, everybody spends 90% of their extra income and saves 10%. How much difference would that make to the amount of extra income generated?

12. A ball is dropped from a height of 10 feet and bounces on the floor. Each time it bounces, it rebounds to 3/4 of the height of the previous bounce.

(a) Write an expression for the height to which the ball rises after it hits the floor for the 10th time and also for the n-th time.

(b) Write an expression for the total distance the ball has traveled when it hits the ground for the second time. Also for the 10th time and the n-th time.

(c) Explain why the total distance the ball has traveled approaches a finite limit as the number of bounces increases without bound. Compute this limit.

13. An object dropped from a height h feet above the ground and subject only to gravity takes \(\sqrt{\frac{2h}{g}}\) seconds to hit the ground, where \(g \approx 32.2\) ft/sec².
is the acceleration due to gravity. Use this formula to show that the total time taken by the ball in the previous problem is finite, even though the number of bounces is infinite. Find this total time. (Note that the amount of time taken by the ball to rise to a given height is the same as the time the ball takes to fall from that height.)

4.2 The Comparison Test and Some Properties of Series

We recall that the question of convergence or non-convergence of sequences is much simpler for monotonic sequences. The sequence of partial sums of an infinite series will be monotonic exactly when all the terms in the infinite series have the same sign. Fortunately it is possible to base most of the theory of infinite series on this simple situation.

It is sufficient to consider the case when all terms are non-negative. Then the sequence of partial sums is non-decreasing. By Theorem 50 a non-decreasing sequence of partial sums converges if and only if it is bounded above. Thus to establish convergence of an infinite series with non-negative terms it is sufficient to show that all partial sums are bounded above by some fixed upper bound. This then insures that the partial sums ”pile up” at a limiting value – their least upper bound. For the most part, this is done by comparing the series to a known series. The following theorem, known as the **Comparison Test** is the basic result.

**Theorem 54** Let \( \{a_k\}_{k=1}^{\infty} \) and \( \{b_k\}_{k=1}^{\infty} \) be two sequences of real numbers such that for each \( k \), \( 0 \leq a_k \leq b_k \). Then

1. if \( \sum_{k=1}^{\infty} b_k \) converges, then \( \sum_{k=1}^{\infty} a_k \) converges;
2. if \( \sum_{k=1}^{\infty} a_k \) diverges, then \( \sum_{k=1}^{\infty} b_k \) diverges.

**Proof.** Let \( S_n = \sum_{k=1}^{n} a_k, T_n = \sum_{k=1}^{\infty} b_k \). Then for each \( n \), \( S_n \leq T_n \) because each is the sum of \( n \) terms and each term in \( T_n \) is greater than or equal to the corresponding term in \( S_n \). Recall that each of the sequences \( \{S_n\} \) and \( \{T_n\} \) is non-decreasing. If \( \sum_{k=1}^{\infty} b_k \) converges, then \( T = \lim_{n \to \infty} T_n \) exists. Since \( T \) is in fact the least upper bound of the \( T_n \), \( T_n \leq T \) for each \( n \). But then also

\[
S_n \leq T_n \leq T
\]

for each \( n \), that is, the sequence \( \{S_n\} \) is bounded above by \( T \). Thus By Theorem 50, \( \lim_{n \to \infty} S_n = \sum_{k=1}^{\infty} a_k \) exists.

Suppose on the other hand that \( \sum_{k=1}^{\infty} a_k \) diverges. Then \( \sum_{k=1}^{\infty} b_k \) must also diverge, since otherwise \( \lim_{n \to \infty} T_n = T \) exists, and we have already seen that this would force \( \sum_{k=1}^{\infty} a_k \) to converge.
Examples. The most straightforward uses of the Comparison Test are to show convergence of series by comparison with a geometric series or to show divergence by comparison with the harmonic series. Here are two simple examples.

1. Consider
\[ \sum_{k=1}^{\infty} \frac{1}{2^k + k + 1}. \]
For each \( n \), \( \frac{1}{2^k + k + 1} < \frac{1}{2^k} \). Moreover, \( \sum_{k=1}^{\infty} 2^{-k} \) converges with sum equal to 1 (note that the first term is \( \frac{1}{2} \), not 1.) Thus the smaller series also converges to a sum less than 1.

2. On the other hand, \( \frac{1}{\sqrt{k}} \geq \frac{1}{k} \) for each \( k \geq 1 \). Thus \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \) diverges.

Remark. Notice that the first comparison to a geometric series gives us only limited information about the sum of the smaller series — the sum is less than 1. To determine it more precisely would require additional work. However we have established that there is something to look for — the series does converge.

We can improve our ability to compare infinite series by establishing some general properties of series. These work for any series of real numbers; it is not necessary that the terms be positive.

**Theorem 55** If \( \{a_k\}_{k=1}^{\infty} \) and \( \{b_k\}_{k=1}^{\infty} \) are sequences of real numbers such that \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) converge, then \( \sum_{k=1}^{\infty} (a_k + b_k) \) converges and
\[
\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.
\]
If \( \sum_{k=1}^{\infty} a_k \) converges and \( c \) is a real number, then \( \sum_{k=1}^{\infty} ca_k \) converges and
\[
\sum_{k=1}^{\infty} ca_k = c \sum_{k=1}^{\infty} a_k.
\]

**Proof.** These properties follow immediately from parts (a) and (b) of Theorem 43 by writing them as statements about the corresponding sequences of partial sums. The details are left as exercises.

A further corollary establishes the important principle that the convergence or divergence of an infinite series depends only on the "tail" of the series — the behavior of the first finite number of terms does not affect convergence.

**Corollary 56** The infinite series \( \sum_{k=k_0}^{\infty} a_k \) converges if and only if for some \( l > k_0 \), \( \sum_{k=l}^{\infty} a_k \) converges. If so,
\[
\sum_{k=k_0}^{\infty} a_k = \sum_{k=k_0}^{l-1} a_k + \sum_{k=l}^{\infty} a_k.
\]
and, more generally, for each \( m \geq k_0 \), \( \sum_{k=m}^{\infty} a_k \) converges and
\[
\sum_{k=k_0}^{\infty} a_k = \sum_{k=k_0}^{m-1} a_k + \sum_{k=m}^{\infty} a_k.
\]

**Proof.** Suppose that \( \sum_{k=l}^{\infty} a_k \) converges. Define
\[
b_k = \begin{cases} 
0 & \text{if } k < l, \\
a_k & \text{if } k \geq l,
\end{cases} \quad c_k = \begin{cases} 
a_k & \text{if } k < l, \\
0 & \text{if } k \geq l.
\end{cases}
\]
Then \( \sum_{k=k_0}^{\infty} b_k \) converges by hypothesis, and \( \sum_{k=k_0}^{\infty} c_k \) converges because only the first \( l-k_0-1 \) terms can be different from zero. Moreover, \( a_k = b_k + c_k \) for each \( k \). Thus it follows from the previous Corollary that \( \sum_{k=k_0}^{m} a_k \) converges.

For the other direction, for any \( m \geq k_0 \), define \( c_k = -a_k \) for \( k = k_0, \ldots, m-1, = 0 \) for \( k \geq m \), and \( b_k = a_k \) for each \( k \) so that
\[
b_k + c_k = \begin{cases} 
0 & \text{if } k < m, \\
a_k & \text{if } k \geq m.
\end{cases}
\]
Thus \( \sum_{k=k_0}^{\infty} a_k \) converging implies that \( \sum_{k=m}^{\infty} a_k \) converges for each \( m \geq k_0 \).

Theorem 58 Suppose that \( \sum_{k=k_0}^{\infty} a_k \) and \( \sum_{k=k_0}^{\infty} b_k \) are two series of non-negative terms such that
\[
\lim_{k \to \infty} \frac{a_k}{b_k} = L
\]
exists.

(1) If \( L > 0 \), then \( \sum_{k=k_0}^{\infty} b_k \) converges if and only if \( \sum_{k=k_0}^{\infty} a_k \) converges.
(2) If \( L = 0 \) and \( \sum_{k=k_0}^{\infty} b_k \) converges, then \( \sum_{k=k_0}^{\infty} a_k \) converges.
Proof. Suppose that $\sum_{k=k_0}^\infty b_k$ converges. By Corollary 56, in order to show that $\sum_{k=k_0}^\infty a_k$ converges, it suffices to show that $\sum_{k=l}^\infty a_k$ converges for some integer $l$. If the limit exists, then there is an integer $K$ such that

$$|L - \frac{a_k}{b_k}| < 1$$

for each $k > K$, that is, such that

$$\frac{a_k}{b_k} < L + 1$$

for each $k > K$. But then for each $N > K + 1$,

$$\sum_{k=K+1}^N a_k < (L + 1) \sum_{k=K+1}^N b_k \leq (L + 1) \sum_{k=K+1}^\infty b_k. \quad (4.1)$$

Since this provides an upper bound for all of the partial sums, $\sum_{k=K+1}^\infty a_k$ converges.

If $L > 0$, and if $\sum_{k=k_0}^\infty a_k$ converges, then for some $K$,

$$|L - \frac{a_k}{b_k}| < \frac{L}{2}$$

or $\frac{a_k}{b_k} > \frac{L}{2}$. Then a similar argument gives that for all $N$ greater than some $K$,

$$\sum_{k=K+1}^N b_k < \frac{2}{L} \sum_{k=K+1}^N a_k \leq \sum_{k=K+1}^\infty a_k.$$

Thus $\sum_{k=K+1}^\infty b_k$ converges.

This result can be very handy in reducing the consideration of infinite series to considering the convergence of the associated series of “most important part.”

Examples. 1. $\sum_{k=1}^\infty \frac{1}{k^2 - 17k + 23}$ converges since $\sum_{k=1}^\infty \frac{1}{k^2}$ converges and

$$\lim_{k \to \infty} \frac{1}{k^2 - 17k + 23} = \lim_{k \to \infty} \frac{k^2}{k^2 - 17k + 23} = 1.$$

Note that since $k^2 - 17k + 23 < k^2$ for $k > 1$, this result does not follow in a trivial way from the Comparison Theorem.

2. Similarly, $\sum_{k=1}^\infty \frac{k}{k^2 + 19}$ diverges because $\sum_{k=1}^\infty \frac{1}{k}$ diverges and

$$\lim_{k \to \infty} \frac{k}{k^2 + 19} = 1.$$
EXERCISES.

Test for convergence or divergence by comparison with a suitable geometric series. If the series converges, estimate the sum.

1. $\sum_{k=0}^{\infty} \frac{1}{2^k + 1}$
2. $\sum_{k=1}^{\infty} \frac{1}{2^k - 1}$
3. $\sum_{k=0}^{\infty} \frac{k}{2^k}$
4. $\sum_{k=0}^{\infty} \frac{k^2}{2^{k/2}}$

5. Prove the assertion about sums in Theorem 55.
6. Prove the assertion about constant multiples of series in Theorem 55.
7. Let $\sum_{k=1}^{\infty} a_k$ be a convergent series of non-negative terms. Define $\sum_{k=0}^{\infty} b_k$ by $b_0 = a_1, b_1 = a_2, b_2 = a_3 + a_4,$ and in general $b_k = a_{2k-1} + \ldots + a_{2k}$ for each $k \geq 2$. Prove that $\sum_{k=0}^{\infty} b_k$ converges and that $\sum_{k=1}^{\infty} a_k = \sum_{k=0}^{\infty} b_k$. Suggestion: Adapt the proof of Theorem 54. to show $\sum_{k=0}^{\infty} b_k \leq \sum_{k=1}^{\infty} a_k$. For equality you will need to use the definition of convergence with care.

4.3 A Test for Series with Exponential Function-like Terms

A geometric series $\sum_{k=0}^{\infty} ar^k$ can be thought of as a series of the form $\sum_{k=0}^{\infty} f(k)$ where $f$ is the exponential function $f(x) = ar^x$ with constant ratio $r$. (Recall that the defining property of an exponential function is that each time you increase the independent variable by one, the value of the function is multiplied by a constant ratio.) Many series have terms that behave in a somewhat similar way and which can therefore be shown to converge or diverge by comparison with a geometric series. The following much used test for convergence, the Ratio Test, is just an institutionalized comparison with a geometric series.

**Theorem 59** Let $\sum_{k=0}^{\infty} a_k$ be a series of non-negative terms. Suppose that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = R$ exists. Then

$$\sum_{k=0}^{\infty} a_k \begin{cases} \text{converges} & \text{if } R < 1, \\ \text{diverges} & \text{if } R > 1, \\ ? & \text{if } R = 1. \end{cases}$$

Here the last row means that no conclusion can be drawn if $R = 1$.

**Proof.** Suppose that $R < 1$. The assumption means that eventually the ratio $\frac{a_{k+1}}{a_k}$ is nearly equal to $R$ and so is also less than 1. We know that a geometric series with common ratio $R$ converges. We cannot quite compare our series with such a series, because the ratios $\frac{a_{k+1}}{a_k}$ may be a little more than $R$. However, if we give a little bit away, we can compare our series to one with the slightly larger common ratio $\frac{R+1}{2}$.
More precisely, there is $K$ such that for $k > K$,
\[
\left| R - \frac{a_{k+1}}{a_k} \right| < \frac{1 - R}{2}.
\]

For these values of $k$,
\[
a_{k+1} < R + \frac{1 - R}{2} = \frac{1 + R}{2} < 1.
\]
Thus $a_{K+2} < \frac{1+R}{2} a_{K+1}, a_{K+3} < \frac{1+R}{2} a_{K+2} < \left( \frac{1+R}{2} \right)^2 a_{K+1}$, and in general, for any positive integer $k > K + 1$,
\[
a_k < \left( \frac{1 + R}{2} \right)^{k-(K+1)} a_{K+1}.
\]

Thus $\sum_{k=K+1}^{\infty} a_k$ converges by comparison with the geometric series
\[
a_{K+1} + \left( \frac{1 + R}{2} \right) a_{K+1} + \left( \frac{1 + R}{2} \right)^2 a_{K+1} + \ldots = a_{K+1} \left( 1 + r + r^2 + \ldots \right)
\]
with $r = \frac{1+R}{2}$. By Corollary 56, this proves that $\sum_{k=k_0}^{\infty} a_k$ converges.

If $R > 1$, then the geometric series with common ratio $\frac{1+R}{2}$ diverges, and a similar comparison argument shows that $\sum_{k=k_0}^{\infty} a_k$ diverges.

**Remark.** To see that no conclusion can be drawn if $R = 1$, note that for any $p > 0$,
\[
\lim_{k \to \infty} \frac{1}{(k+1)^p} = \lim_{k \to \infty} \frac{k^p}{(k+1)^p} = 1.
\]
But we already know that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

**EXERCISES.**

Test for convergence using the Ratio Test. If no conclusion can be drawn, try to use a comparison test instead.

1. $\sum_{k=2}^{\infty} \frac{1}{2^k - 3}$
2. $\sum_{k=1}^{\infty} \frac{1}{k^2}$
3. $\sum_{k=1}^{\infty} \frac{k}{2^k}$
4. $\sum_{k=1}^{\infty} \frac{4^k}{3^{2k-1}}$
5. $\sum_{k=1}^{\infty} \frac{(k+1)^2}{k2^k}$
6. $\sum_{k=2}^{\infty} \frac{k^3 + 1}{k^4 - 1}$
7. $\sum_{k=1}^{\infty} \frac{3^k k^2}{k!}$
8. $\sum_{k=1}^{\infty} e^{1/k}$

9. Write out the details of the proof that if $R > 1$, then $\sum_{k=k_0}^{\infty} a_k$ diverges by comparison with a geometric series with common ratio $\frac{1 + R}{2}$.

10. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of non-negative numbers.
   (a) Prove that if for some $K < 1, a_k^{1/k} \leq K$ for each $k$, then $\sum_{k=k_0}^{\infty} a_k$ converges.
   (b) Prove that if $\lim_{k \to \infty} a_k^{1/k} < 1$, then $\sum_{k=k_0}^{\infty} a_k$ converges.

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4.4 Two Tests for Series with Power Function-like Terms

A second important class of infinite series are those of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$, where $p$ is a positive real number (not necessarily an integer). We can think of these as having the general form $\sum_{k=1}^{\infty} f(k)$ where $f$ is the power function $f(x) = x^{-p}$. Here the terms approach zero much more slowly than the terms of a geometric series. For instance, both $\left(\frac{1}{2}\right)^k \to 0$ as $k \to \infty$ and $\frac{1}{k^p} \to 0$ as $k \to \infty$, but terms in the first sequence get smaller much more quickly. (When $k = 10$, $\left(\frac{1}{2}\right)^{10} \approx 0.001$ while $\frac{1}{10^{10}} = 0.01$. When $k = 20$, $\left(\frac{1}{2}\right)^{20} \approx 10^{-6}$ while $\frac{1}{20^{20}} = 0.0025$. When $k = 30$, $\left(\frac{1}{2}\right)^{30} \approx 10^{-9}$ while $\frac{1}{30^{30}} \approx 0.0011$.) This is just a variation on the familiar fact that increasing exponential functions increase much more rapidly than polynomial functions—if you take reciprocals in that statement you get that decreasing exponential functions get very close to zero much sooner than negative power functions.

The Ratio Test is not useful for such series, since the common ratio $\frac{a_{k+1}}{a_k}$ approaches 1, the indeterminate case in the test. So we need a test able to make finer distinctions. We will look at two such tests, both restricted to series with decreasing terms.

The first is the Cauchy Condensation Test.

**Theorem 60** Suppose that $a_1 \geq a_2 \geq \ldots \geq 0$. For $k = 0, 1, 2, \ldots$, set $b_k = 2^k a_{2^k}$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges. If they converge, then

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k \leq 2 \sum_{k=1}^{\infty} a_k.$$

**Proof.** We use the fact that for both series the sequence of partial sums is non-decreasing, and so converges if and only if it is bounded above. We have $b_0 = a_1$. Then, since the sequence $\{a_k\}_{k=1}^{\infty}$ is non-increasing, $b_0 = a_1, b_1 = 2a_2 \geq a_2 + a_3$ and more generally $b_k = 2^k a_{2^k} \geq \sum_{j=2^k}^{2^{k+1}-1} a_j$ since each of the $2^k$ terms in the sum is at most equal to $a_{2^k}$. Thus if $\sum_{k=0}^{\infty} b_k$ converges, we have for each $m$

$$\sum_{k=0}^{\infty} b_k \geq \sum_{k=0}^{m} b_k \geq \sum_{j=1}^{2^{m+1}-1} a_j.$$

This says that all the partial sums $S_n$ of $\sum_{j=1}^{\infty} a_j$ where $n$ has the form $n = 2^{m+1} - 1$ are bounded by the fixed number $\sum_{k=0}^{\infty} b_k$. Since the sequence of partial sums is
non-decreasing, \( S_n \leq \sum_{k=0}^{\infty} b_k \) for all \( n \). Thus \( \sum_{k=1}^{\infty} a_k \) converges and \( \sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k \).

For the other direction, suppose that \( \sum_{k=1}^{\infty} a_k \) converges. We have \( b_0 = a_1 \leq 2a_1, b_1 = 2a_2, b_2 = 4a_4 \leq 2(a_3 + a_4) \), and in general
\[
b_k = 2^k a_{2k} \leq 2(a_{2k-1} + a_{2k-1} + \ldots + a_2)
\]
where we note that the last sum contains \( 2^{k-1} \) terms. Thus for each \( n \), the \( n \)-th partial sum \( T_n \) of \( \sum_{k=0}^{\infty} b_k \) satisfies
\[
T_n = \sum_{k=0}^{n} b_k \leq 2 \sum_{k=1}^{2^n} a_k \leq 2 \sum_{k=1}^{\infty} a_k \quad (4.2)
\]
and again we can conclude that \( \sum_{k=0}^{\infty} b_k \) converges and \( \sum_{k=0}^{\infty} b_k \leq 2 \sum_{k=1}^{\infty} a_k \).

Examples. 1. Let \( a_k = \frac{1}{k} \). Then \( \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k} \) is the harmonic series. We have \( b_k = 2^k \frac{1}{2^k} = 1 \) for each \( k \). Thus \( \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} 1 \) which certainly diverges since the partial sums are just \( \sum_{k=0}^{n} 1 = n + 1 \). We see again that the harmonic series diverges.

2. Let \( a_k = \frac{1}{k^2} \). Then \( \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \) and \( b_k = 2^k \frac{1}{(2^k)^2} = \frac{1}{2^k} \). This time \( \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \frac{1}{2^k} \) is a convergent geometric series with sum 2. Thus \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges to a sum less than 2. (It turns out that \( \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.6449 \) but we are not in a position to prove this here.)

3. More generally, let \( a_k = \frac{1}{k^p} \). Then \( \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p} \) and \( b_k = 2^k \frac{1}{(2^k)^p} = \frac{1}{2^{k(p-1)}} \). If \( p > 1 \), \( \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \frac{1}{2^{k(p-1)}} = \frac{2^{p-1}}{2^{p-1} - 1} \) where the sum comes from the formula for the sum of a geometric series with initial term \( a = 1 \) and common ratio \( r = 1/2^{p-1} < 1 \). If \( p \leq 1 \), then each \( b_k \geq 1 \) so that \( \sum_{k=0}^{n} b_k \geq n + 1 \) and the series diverges. We conclude that
\[
\sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges if and only if } p > 1.
\]
Note that this formula applies to all real numbers \( p > 1 \), not just integers, and that from the right hand part of inequality \( (4.2) \)
\[
\sum_{k=1}^{\infty} \frac{1}{k^p} \geq \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^{k(p-1)}} = \frac{2^{p-2}}{2^{p-1} - 1} \rightarrow \infty \text{ as } p \rightarrow 1,
\]
that is, the \( p \)-series is converging more and more slowly to a larger and larger sum as \( p \) decreases and approaches 1.

**Remark.** These examples illustrate the point of the Condensation Test. Taking just scattered terms out of the original series, tends to convert a series with power function-like terms to a series with exponential function-like terms, and these are much easier to investigate.

Another method for investigating series with non-increasing power-like terms is to compare them to improper integrals. This allows use of all the machinery for finding antiderivatives.

Recall from Section 2.3 that if \( a \) is a fixed real number, and \( f \) is a real-valued function that is Riemann integrable on \([a, b]\) for each real number \( b > a \), then the improper integral \( \int_{a}^{\infty} f(x) \, dx \) is defined by

\[
\int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx
\]

if this limit exists. In this case we say the improper integral converges. If the limit does not exist, it diverges.

**Examples.** We have already calculated that

\[
\int_{a}^{\infty} \frac{1}{x^p} \, dx \begin{cases} 
\text{converges to} \frac{1}{p-1} & \text{if} \ p > 1, \\
\text{diverges} & \text{if} \ p \leq 1.
\end{cases}
\]

**Remark.** It should be realized that the function \( f \) in the definition of improper integral need not be non-negative valued. However, for the purposes of comparison with infinite series, it is non-negative valued functions that are of the most use. Note that in this case, the function

\[ F(b) = \int_{1}^{b} f(x) \, dx \]

is a non-decreasing function of \( b \). Thus this function approaches a limit as \( b \to \infty \) if and only if its values are bounded above, and if so, then the limit, \( \int_{1}^{\infty} f(x) \, dx \), is the least upper bound of the numbers \( F(b) \).

The following theorem about comparing improper integrals to infinite series is called the **Integral Test.**

**Theorem 61** Let \( f \) be a positive non-increasing function defined for all \( x \geq a \), for some fixed real number \( a \). Let \( k_0 \) be the smallest integer greater than or equal to \( a \). Then

\[
\int_{a}^{\infty} f(x) \, dx \text{ and } \sum_{k=k_0}^{\infty} f(k)
\]

either both converge or both diverge. More precisely, for any integer \( n > k_0 \),

\[
\sum_{k=k_0+1}^{n} f(k) \leq \int_{k_0}^{n} f(x) \, dx \leq \sum_{k=k_0}^{n-1} f(k).
\]
Proof. The two sums represent respectively the right hand sum and the left hand sum for the integral between them which correspond to the division of the interval \([k_0, n]\) into subintervals of length 1. That they are then related to the integral as in the inequality follows immediately from the fact that the function \(f\) is non-increasing. (Compare the areas in the diagrams below.)

\[
\int_{k_0}^{n} f(x) \, dx \leq \sum_{k = k_0}^{n-1} f(k) \leq \sum_{k = k_0+1}^{n} f(k) \leq \int_{k_0}^{n} f(x) \, dx.
\]

If the improper integral converges, then the left hand inequality shows that for any \(n\),

\[
\sum_{k = k_0+1}^{n} f(k) \leq \int_{k_0}^{n} f(x) \, dx \leq \int_{k_0}^{\infty} f(x) \, dx.
\]

Thus the sequence of partial sums is bounded above, and the series converges. The right hand inequality may be used similarly to show that if the infinite series converges, then the improper integral converges.

Example. From the Integral Test and the example above we see once again that

\[
\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} 
\text{converges} & \text{if } p > 1, \\
\text{diverges} & \text{if } p \leq 1.
\end{cases}
\]

Note that the case \(p = 1\) is the harmonic series that we have already shown to diverge by direct estimation. Note also that the sequence of individual terms in each of these series converges to zero much more slowly than any geometric series, so we could not hope to investigate these series by comparison with a geometric series.

EXERCISES.

1. Investigate these series for convergence or divergence. Try both the Condensation Test and the Integral Test. You may also want to compare with a suitable series with simpler terms.

\[
\begin{align*}
(a) & \quad \sum_{k=2}^{\infty} \frac{1}{k \ln k} \\
(b) & \quad \sum_{k=1}^{\infty} \frac{3}{2k - 1} \\
(c) & \quad \sum_{k=1}^{\infty} \frac{3}{k^2 + 1} \\
(d) & \quad \sum_{k=2}^{\infty} \frac{\ln k}{k} \\
(e) & \quad \sum_{k=1}^{\infty} \frac{3}{k + 1000} \\
(f) & \quad \sum_{k=1}^{\infty} \frac{3}{2k^2 + 7} \\
(g) & \quad \sum_{k=1}^{\infty} \frac{1}{k (k + 1)} \\
(h) & \quad \sum_{k=2}^{\infty} \frac{\ln k}{k^{3/2}} \\
(i) & \quad \sum_{k=1}^{\infty} \frac{\arctan k}{k^2 + 1}
\end{align*}
\]
2. An architect decides to settle the competition for the world’s highest building for all time by designing a building that is infinitely tall. Of course she knows that she has only a finite amount of building materials to work with, so the upper floors will have to be quite small! She decides to have an infinite number of floors in her building, each 10 feet high, with the floor area of the $n$-th floor being a square $10/n$ feet on a side.

(a) Show that the volume of this infinitely high building will be finite. Estimate the volume in cubic feet.

(b) She decides to make all vertical surfaces out of glass. Write an expression for the total area of glass required. Estimate this area.

3. (a) Use the estimates from the proof of the Integral Test to show that

$$\ln (n+1) \leq \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \ln n.$$ 

(b) Suppose that you had started computing partial sums of the harmonic series the day the universe was formed 13 billion years ago and had added one term to the sum every second since then. About how large would the partial sum now be? How large would $n$ be?

The moral of this problem is that just because the partial sums don’t seem to be growing much, you cannot conclude that the series converges.

4. (a) Let $f$ be a positive decreasing function defined for all $x \geq 1$. Use a diagram like the right hand illustration for the proof of the Integral Test to show that the sequence $c_n = \int_1^n f(x) \, dx - (f(2) + f(3) + \ldots + f(n))$ is positive and increasing. Explain how $c_n$ can be interpreted as a ”stack of areas.”

(b) From part (a) or otherwise, show that the sequence

$$d_n = (f(1) + f(2) + \ldots + f(n-1)) - \int_1^n f(x) \, dx$$

is positive and decreasing. Explain how $d_n$ can be interpreted as “what’s left when a stack of areas is removed.”

(c) Prove that $\lim_{n \to \infty} d_n$ exists.

(d) Apply the preceding parts to the function $f(x) = 1/x$. (That is, write out the terms $d_n$ in this case. Be sure to evaluate the integral.) In this case the limit from (c) is called Euler’s constant, usually denoted by $\gamma$. Estimate $\gamma$. 

TWO TESTS FOR SERIES WITH POWER FUNCTION-LIKE TERMS 101
4.5 Test for Divergence; Alternating Series; Absolute Convergence

We turn now to a discussion of infinite series where the terms are not necessarily non-negative. We mention first a simple test that can be used to show divergence, but not convergence.

**Theorem 62 (Test for Divergence)** If the series \( \sum_{k=k_0}^{\infty} a_k \) converges, then \( \lim_{k \to \infty} a_k \) exists and equals zero. Equivalently, if \( \lim_{k \to \infty} a_k \) does not exist, or does not equal zero, then \( \sum_{k=k_0}^{\infty} a_k \) diverges.

**Remark.** It is essential not to confuse this statement with its converse. It is not true that if \( a_k \to 0 \), then \( \sum_{k=k_0}^{\infty} a_k \) converges. The harmonic series is one example of this. This is a very crude test for divergence, but is sometimes helpful.

**Proof.** Suppose that \( \sum_{k=k_0}^{\infty} a_k \) converges, that is, the sequence of partial sums \( S_n = \sum_{k=k_0}^{n} a_k \) converges to a limit \( S \). Let \( \epsilon > 0 \) be given. There is an integer \( N \) so that for \( n > N \), \( |S - S_n| < \epsilon/2 \). Note that \( a_k = S_k - S_{k-1} \). Choose \( K = N + 1 \). Then for \( k > K \),

\[
|a_k - 0| = |a_k| = |S_k - S_{k-1}| = |(S_k - S) + (S - S_{k-1})| \\
\leq |S_k - S| + |S - S_{k-1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Thus \( a_k \to 0 \) as \( k \to \infty \).

The next theorem describes a rather special situation, but one that occurs often enough to merit explicit mention. We first illustrate it with an example.

**Example.** Consider \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \). This series has partial sums

\[
S_1 = 1, \quad S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \quad S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60}, \ldots \\
S_2 = 1 - \frac{1}{2} = \frac{1}{2}, \quad S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}, \quad S_6 = S_5 - \frac{1}{6} = \frac{37}{60}, \ldots
\]

where it is easy to see

\[
\frac{1}{2} < \frac{7}{12} < \frac{37}{60} < \ldots < \frac{47}{60} < \frac{5}{6} < 1,
\]

that is,

\[
S_2 < S_4 < S_6 < \ldots < S_5 < S_3 < S_1.
\]

The alternation in sign of the terms causes the partial sums to jump back and forth, and the fact that the terms are decreasing in absolute value causes this oscillation to have steadily decreasing amplitude, so that the partial sums are nested as shown. But then the increasing sequence \( S_2, S_4, S_6, \ldots \) is bounded.
above (by $S_1$, for instance) and so converges to its least upper bound, $S_L$. Similarly, the decreasing sequence $S_1, S_3, S_5, ...$ is bounded below (by $S_2$) and so converges to its greatest lower bound, $S_R$. Clearly $S_L \leq S_R$. But for any $\epsilon$, if $n$ is even and $n > 1/\epsilon$, then

$$S_n < S_L \leq S_R < S_{n+1}$$

and $S_R - S_L = \frac{1}{n+1} < \epsilon$, so that $S_R - S_L < \epsilon$. Since this is true for any $\epsilon$, $S_L = S_R$ is the limit of the sequence of partial sums, that is, the alternating harmonic series converges.

The following generalization of this example is called the Alternating Series Test.

**Theorem 63** Let $\sum_{k=k_0}^{\infty} a_k$ be an infinite series such that
(1) the signs of the terms $a_k$ are alternately positive and negative, and
(2) the sequence $\{|a_k|\}_{k=k_0}^{\infty}$ is decreasing and converges to zero as $k \rightarrow \infty$. Then $\sum_{k=k_0}^{\infty} a_k$ converges.

**Proof.** Suppose, to be definite, that $a_{k_0} > 0$. Then it is easy to see from properties (1) and (2) that the sequence $\{S_n\}$ of partial sums splits into an increasing part and a decreasing part

$$S_{k_0} < S_{k_0+2} < ... < S_{k_0+3} < S_{k_0+1}.$$ 

We have again that the increasing sequence is bounded above and so converges to its least upper bound $S_L$, and the decreasing sequence is bounded below and converges to its greatest lower bound $S_R$. As in the example, it follows from the assumption that $a_k \rightarrow 0$ as $k \rightarrow \infty$ that $S_L = S_R$, so that the sequence of partial sums converges to their common value, that is, the series $\sum_{k=k_0}^{\infty} a_k$ converges.

**Example.** We return to $\int_0^{\infty} \frac{\sin x}{x} dx$, which was mentioned in Section 2.3 but not shown either to converge or to diverge We will show that this improper integral converges by relating it to an alternating series. Let $a_k = \int_{(k-1)\pi}^{k\pi} \frac{\sin x}{x} dx$, $k = 1, 2, ....$ Then $(-1)^{k-1} a_k > 0$ since $(-1)^{k-1} \sin x > 0$ on $((k-1)\pi, k\pi)$.
Also, from the substitution $u = x - \pi, du = dx$ we get

$$|a_{k+1}| = \left| \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x} \, dx \right| = \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} \, dx$$

$$= \int_{(k-1)\pi}^{k\pi} \frac{|\sin (u + \pi)|}{u + \pi} \, du = \int_{(k-1)\pi}^{k\pi} \frac{|\sin u|}{u + \pi} \, du$$

$$< \int_{(k-1)\pi}^{k\pi} \frac{|\sin u|}{u} \, du = |a_k|.$$  

Here we have been able to move the absolute value signs in and out of the integral because the sign of $\sin x$ is constant on each interval $((k - 1)\pi, k\pi)$.

The diagram below shows how the graph of $\frac{\sin x}{x}$ between $x = 3\pi$ and $x = 4\pi$, when flipped over and translated to the left by $\pi$, fits under the graph of $\frac{\sin x}{x}$ between $x = 2\pi$ and $x = 3\pi$.

Finally,

$$|a_k| \leq \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} \, dx \leq \int_{(k-1)\pi}^{k\pi} \frac{1}{x} \, dx \leq \frac{\pi}{(k-1)\pi} = \frac{1}{k-1} \to 0.$$
Thus $\sum_{k=1}^{\infty} a_k$ is a convergent alternating series, and clearly $\sum_{k=1}^{\infty} a_k = \int_0^{\infty} \frac{\sin x}{x} \, dx$.

Finally, we show how the convergence of an infinite series $\sum_{k=k_0}^{\infty} a_k$ whose terms are not all non-negative may be established by looking at the series $\sum_{k=k_0}^{\infty} |a_k|$.

**Definition 64** The series $\sum_{k=k_0}^{\infty} a_k$ converges absolutely if the series $\sum_{k=k_0}^{\infty} |a_k|$ converges. If $\sum_{k=k_0}^{\infty} a_k$ converges, but $\sum_{k=k_0}^{\infty} |a_k|$ diverges, we say $\sum_{k=k_0}^{\infty} a_k$ converges conditionally.

**Examples.** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges conditionally, since the harmonic series diverges. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ converges absolutely, since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

**Theorem 65** If $\sum_{k=k_0}^{\infty} a_k$ converges absolutely, then $\sum_{k=k_0}^{\infty} a_k$ converges, and

$$\left| \sum_{k=k_0}^{\infty} a_k \right| \leq \sum_{k=k_0}^{\infty} |a_k|.$$ 

**Remark.** Note that there is something to prove here, since we did not assume in the definition of absolute convergence that the series $\sum_{k=k_0}^{\infty} a_k$ itself converges. For the example just given, this amounts to the statement that we can establish the convergence of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ without ever working explicitly with this series, since we already know that the series of absolute values of its terms converges.

Note also that the harmonic series/alternating harmonic series examples show that the converse of this statement is not true, that is, the situation of conditional convergence does actually occur.

**Proof.** Suppose that $\sum_{k=k_0}^{\infty} |a_k|$ converges. We have that

$$0 \leq \frac{1}{2} (|a_k| - a_k) \leq |a_k|, \quad 0 \leq \frac{1}{2} (|a_k| + a_k) \leq |a_k|.$$ 

Thus by the Comparison Theorem we have that

$$\sum_{k=k_0}^{\infty} \frac{1}{2} (|a_k| - a_k) \quad \text{and} \quad \sum_{k=k_0}^{\infty} \frac{1}{2} (|a_k| + a_k) \quad \text{converge.}$$ 

Moreover,

$$\frac{1}{2} (|a_k| + a_k) + (-1) \frac{1}{2} (|a_k| - a_k) = a_k.$$

Then

$$\sum_{k=k_0}^{\infty} a_k = \sum_{k=k_0}^{\infty} \frac{1}{2} (|a_k| + a_k) + (-1) \sum_{k=k_0}^{\infty} \frac{1}{2} (|a_k| - a_k)$$
converges by the result on sums and multiples of convergent series. Moreover, since for each \( n \),
\[
\left| \sum_{k=1}^{n} a_k \right| \leq \sum_{k=1}^{n} |a_k| \leq \sum_{k=1}^{\infty} |a_k| ,
\]
it follows that
\[
\left| \sum_{k=1}^{\infty} a_k \right| = \lim_{n \to \infty} \left| \sum_{k=1}^{n} a_k \right| \leq \sum_{k=1}^{\infty} |a_k| .
\]

**Remark.** A conditionally convergent series like \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \) has a very strange and non-sumlike property. We are accustomed to the fact that when we add numbers, the order with which we add them does not affect the sum. This property continues to hold for absolutely convergent series, but not for conditionally convergent ones. The terms of any conditionally convergent series can be rearranged to converge to any real number whatever. I will illustrate that with \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \) after first describing what is behind the construction

Let \( P_n \) denote the sum of the first \( n \) terms in the alternating harmonic series: \( P_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \), and let \( N_n \) denote the sum of the absolute values of the first \( n \) negative terms: \( N_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} \). Then, since we can certainly rearrange finite sums,
\[
S_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \left( 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \right) - \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} \right) = P_n - N_n.
\]

Now \( N_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} = \frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right) \) is just half the \( n \)-th partial sum of the harmonic series. Thus \( N_n \to \infty \) as \( n \to \infty \). Since \( P_n \) has the same number of terms as \( N_n \) and corresponding terms are larger, also \( P_n \to \infty \) as \( n \to \infty \). The reason that \( S_{2n} \) can approach a finite limit \( S_\infty \) is that the very large numbers \( P_n \) and \( N_n \) nearly cancel each other out, and if we tried to fit this into the context of Theorem 55 (which we can’t) it would come out looking like
\[
S_\infty = \lim_{n \to \infty} S_{2n} = \lim_{n \to \infty} (P_n - N_n) = \lim_{n \to \infty} P_n - \lim_{n \to \infty} N_n = \infty - \infty.
\]
(An exercise below will ask you explain which step in this chain of equalities is incorrect.)

You can think of this as being rather like the behavior of the balance in your checking account. You make deposits (positive terms) and write checks (negative terms), and the sum of these up to the present is your current balance. Over a long period the sum of your deposits alone will be very large, and the
sum of your checks will also be very large, but if you manage your money well, they will more or less cancel out and your balance will stay positive, though perhaps small.

We can use this cancellation of large quantities to rearrange the terms of the alternating harmonic series (but without changing any negative signs to positive or vice versa) to produce any desired sum whatever. As an example, suppose we want a sum of 10. Construct a new series with the same terms as the alternating harmonic series (but not in the same order) as follows: at the beginning of the new series place enough of the positive terms to give a sum between 10 and 11. (Since \( P_n \to \infty \), this is possible.) Now put in enough negative terms to reduce the sum to between 9 and 10. Now add some more positive terms to bring the sum up to between 10 and 10.5. Then enough negative terms to reduce the sum to between 9.5 and 10. Then positive terms to bring the sum to between 10 and 10.25; then negative terms to reduce it to between 9.75 and 10 and so forth. The partial sums will waver back and forth but with decreasing amplitude, and will approach 10. The key to the construction is Theorem 56 which assures us that since the sum of all positive terms diverges, no matter how many positive terms we have removed from the front of the series, what remains is still a divergent series. (And similarly for the negative terms.) Thus no matter how many terms we have used up, we can still find enough left to increase or decrease the partial sum by the amount needed for the construction. To get a sum other than 10, just adjust the first step and then proceed in a similar way.

**EXERCISES.**

Test for convergence or divergence using whatever method seems most convenient.

1. \( \sum_{k=2}^{\infty} \frac{1}{\ln k} \)
2. \( \sum_{k=1}^{\infty} \frac{3 + \sin k}{k^2} \)
3. \( \sum_{k=0}^{\infty} \frac{k^2}{3^k} \)
4. \( \sum_{k=1}^{\infty} \left( \frac{k}{3k+1} \right)^k \)
5. \( \sum_{k=2}^{\infty} \frac{1}{k (\ln k)^2} \)
6. \( \sum_{k=1}^{\infty} \sin \left( \frac{1}{k} \right) \)
7. \( \sum_{k=1}^{\infty} \frac{1}{\pi^k - k^\pi} \)
8. \( \sum_{k=2}^{\infty} \frac{1}{\ln (3k)} \)
9. \( \sum_{k=1}^{\infty} \left( 1 - \cos \frac{\pi}{n} \right) \)

Suggestion for #9: Remember #4 in section 2.6.

Test for absolute convergence, conditional convergence, or divergence.

10. \( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \sqrt{k}} \)
11. \( \sum_{k=1}^{\infty} \frac{(-3)^k}{k^3} \)
12. \( \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \)
13. \( \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \)
14. \( \sum_{k=1}^{\infty} \frac{\cos (k \pi)}{k^2 + 4k} \)
15. \( \sum_{k=1}^{\infty} \frac{(2k)^k}{k^{2k}} \)
16. \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sqrt{k}}{3k + 7} \)
17. \( \sum_{k=2}^{\infty} \frac{-k}{k^2 + 1} \)
18. \( \sum_{k=1}^{\infty} (-1)^k \ln \frac{1}{k} \)
19. \( \sum_{k=1}^{\infty} \frac{\sin k}{k^{1.01}} \)
20. \( \sum_{k=1}^{\infty} \frac{1}{k^{1/2} + k^{3/2}} \)
21. \( \sum_{k=1}^{\infty} \frac{(-1.01)^k}{k!} \)

TEST FOR DIVERGENCE; ALTERNATING SERIES; ABSOLUTE CONVERGENCE
22. \(\sum_{m=1}^{\infty} \frac{\sqrt{m+1} - \sqrt{m}}{\sqrt{m}}\)
23. \(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1 \times 3 \times 5 \times \ldots \times (2n-1)}{2 \times 4 \times 6 \times \ldots \times (2n)}\)

24. Show that the word “monotonically” is an essential part of the Alternating Series Test by making up an example of an alternating series \(\sum_{k=1}^{\infty} a_k\) (that is, \(a_k > 0\) when \(k\) is odd and \(a_k < 0\) when \(k\) is even) such that \(a_k \to 0\) as \(k \to \infty\), but the series diverges.

25. Find a specific example of an infinite series with all of these properties:
\(\sum_{k=1}^{\infty} a_k = 2, \sum_{k=1}^{\infty} |a_k| < \infty, \sum_{k=1}^{\infty} a_k\) has infinitely many partial sums greater than 2 and infinitely many partial sums less than 2. (Suggestion: One method is to construct the partial sums first and go backwards to the terms.)

26. Which equality in the string \(S_{\infty} = \ldots = \infty - \infty\) on the previous page is not justified? Explain.

27. In contrast to conditionally convergent series, the sum of an absolutely convergent series remains the same if the terms are rearranged. (This means that the rearranged sum still converges absolutely and has the same sum as the original series.) Prove this statement for series with non-negative terms by filling in the details of the following argument. Let \(\sum_{k=1}^{\infty} a_k\) be convergent with each \(a_k \geq 0\) and let \(\sum_{k=1}^{\infty} b_k\) be a rearrangement of \(\sum_{k=1}^{\infty} a_k\). The partial sums of \(\sum_{k=1}^{\infty} b_k\) are an increasing sequence that is bounded above by \(\sum_{k=1}^{\infty} a_k\). (Why?) Thus \(\sum_{k=1}^{\infty} b_k\) also converges and \(\sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty} a_k\). Then a similar argument shows that \(\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k\).

4.6 Summary: How to Choose a Test

Given an infinite series of real numbers, \(\sum_{k=k_0}^{\infty} a_k\) what sequence of steps should you go through to determine convergence or divergence efficiently? This will depend on individual circumstances, but here is one reasonable sequence of questions to ask.

(1) Do the terms \(a_k\) approach 0 in magnitude as \(k \to \infty\)? This is the Test for Divergence. If the answer is no, then the series diverges. If it is yes, then
it could either converge or diverge.

(2) For series whose terms are not all of one sign it is better to consider absolute convergence before convergence, because if the series does converge absolutely, then the question of conditional convergence never comes up, while if you determine that it converges (using the Alternating Series Test, for instance) then you still have to worry about absolute convergence.

(3) Which test for absolute convergence should you try first? This depends on the size (and relative size) of the terms.

(i) If they appear to be approaching 0 faster than any power of $k$ (exponentially for instance), then the Ratio Test may work and probably is a good first choice.

(ii) If the terms are approaching 0 like a power of $k$, then the Ratio Test will probably be inconclusive and is not a good choice. Comparison with a series \( \sum_{k=1}^{\infty} \frac{1}{k^p} \) may be the simplest thing to do. If an antiderivative is obvious, the Integral Test may work. The Cauchy Condensation Test is the other obvious method to try.

(4) Only if the terms of the series are not all of the same sign and the series does not converge absolutely but the terms do approach 0 as $k \to \infty$ do you need to think about conditional convergence. The only test we have is the Alternating Series Test. If this does not work, or if the terms do not alternate in sign, then you are on your own.

(5) If none of the tests seem to work, you can try actually computing a sequence of partial sums and looking at the trend to try to decide what is true. This may give you some idea of what to try next, or some chance of discovering that you made a mistake in applying one or more tests. Do not try to do this by hand, you can’t compute enough terms and are too likely to make arithmetical errors. Use a calculator (the SUM and SEQ commands are useful on the TI calculators) or computer. And remember how slowly the harmonic series diverges! That is why this may be a more useful technique for series where the terms change signs in some regular way, so that you may be able to find a pattern in the behavior of the partial sums.
5. POWER SERIES

5.1 Taylor Polynomials

Consider the problem of approximating a function \( f \) near a point \( a \) in its domain by a polynomial. The case of a linear polynomial is familiar – the best linear approximation to \( f \) near \( a \) is given graphically by the line tangent to the graph of \( f \) at \( x = a \). For instance for \( f(x) = e^x \) near \( a = 0 \) we have

\[ e^x \approx 1 + x \]

since here \( f(0) = f'(0) = 1 \).

Suppose we wanted to try to improve the polynomial approximation – that is, get values closer to those of \( e^x \) for a given value of \( x \) or get a polynomial that stays relatively close to \( e^x \) over a larger interval. It is natural to try first for improvement with a quadratic polynomial:

\[ e^x \approx 1 + x + c_2x^2 \] (5.1)

for some best choice of \( c_2 \). How do we choose \( c_2 \)? We could try to determine it experimentally by comparing \( e^x - 1 - x \) to \( x^2 \), seeing whether these quantities appear to be at least approximately proportional near \( 0 \) and if so taking \( c_2 \) to be the constant of proportionality. (We have just subtracted \( 1 + x \) from each side of (5.1)). Equivalently, we could divide again by \( x^2 \) and see whether

\[ C_2(x) = \frac{e^x - 1 - x}{x^2} \] (5.2)

![Figure 5-1 Best linear approximation to \( e^x \) at \( x = 0 \)]
approaches a limit as \( x \to 0 \). Here is a short table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( C_2(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.51709</td>
</tr>
<tr>
<td>0.01</td>
<td>0.50167</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.49834</td>
</tr>
<tr>
<td>0.001</td>
<td>0.50017</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.49983</td>
</tr>
</tbody>
</table>

From the table it seems apparent that \( c_2 = \frac{1}{2} \) is a good choice.

Now let’s try one more step: find \( c_3 \) so that \( e^x \approx 1 + x + \frac{1}{2}x^2 + c_3x^3 \) or equivalently, find the limit of

\[
C_3(x) = \frac{e^x - 1 - x - \frac{1}{2}x^2}{x^3}
\]

as \( x \to 0 \). This time the table looks like this:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( C_3(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.17091</td>
</tr>
<tr>
<td>0.01</td>
<td>0.16708</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.16625</td>
</tr>
<tr>
<td>0.001</td>
<td>0.16670</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.16663</td>
</tr>
</tbody>
</table>

and it seems that \( c_3 = \frac{1}{6} \).

To see the effect of adding these terms to the polynomial approximation to \( e^x \) graphically, consider the following

From these numerical and graphical experiments, it seems reasonable that near a point it is possible to approximate a function such as \( e^x \) closely near a point by a polynomial, and that the higher the degree of the polynomial the closer the approximation. (Provided the polynomial’s coefficients are chosen properly, that is!) Thus it makes sense to try to find a more systematic way of determining for a given function what the approximating polynomials are.

To see how we might do this, let’s compute the successive derivatives at \( x = 0 \) of the original function \( e^x \) and the approximating polynomials. The
results are given in the following chart.

<table>
<thead>
<tr>
<th></th>
<th>$f(x)$</th>
<th>$f(0)$</th>
<th>$f'(0)$</th>
<th>$f''(0)$</th>
<th>$f'''(0)$</th>
<th>$f^{(4)}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^x$</td>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$1 + x + \frac{x^2}{2}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$1$</td>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We see from this that the successively better polynomials match the derivatives of $e^x$ at $x = 0$ for successively more steps. (Of course, as polynomials each will have all derivatives equal to 0 from some point on.)

Thus encouraged, we try to see how to determine a polynomial with prescribed derivatives at $x = 0$. If $p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$, then it is easy to see by computing the successive derivatives and evaluating at $x = 0$ that

$$p(0) = c_0, \quad p'(0) = c_1, \quad \text{and} \quad p''(0) = 2c_2.$$

For instance, $p''(x) = 2c_2 + 3 \cdot 2c_3 x + \ldots + n (n - 1) c_n x^{n-2}$ and substituting $x = 0$ leaves only the constant term. In general, $p^{(k)}(x) = k (k - 1) (k - 2) \ldots 2c_k + \text{(terms involving powers of } x) = k! c_k$ (terms involving powers of $x$), so

$$p^{(k)}(0) = k! c_k$$

where $k!$ is the usual factorial symbol: $k! = k (k - 1) (k - 2) \ldots 3 \cdot 2 \cdot 1$.

Thus matching the first $n$ derivatives at $x = 0$ of a function $f$ with a polynomial $p_n$ of degree $n$ is simple. We simply choose $p_n$ so $c_k = f^{(k)}(0)/k!$, that is,

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n. \quad (5.3)$$

Applying this procedure to $f(x) = e^x$ will produce the coefficients of the approximating polynomial we constructed above.

By a similar procedure, we can determine the polynomial $p$ which has the same value and the same first $n$ derivatives at an arbitrary point $x = a$ as an n-times differentiable function $f$. It is convenient now to think of $p$ as a sum of powers of $x - a$ since then, just as above, each successive derivative of $p$, evaluated at $x = a$ depends on only a single coefficient in the polynomial. The result is that now

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (5.4)$$

**Definition 66** If $f$ is an n-times differentiable function defined for all $x$ near a point $a$, the polynomial (5.4) is called the **n-th Taylor polynomial of $f$ based at $a$**.
We will investigate in section 5.3 just how well \( p_n(x) \) approximates \( f(x) \) for \( x \) in a small interval centered at \( x = a \). We would hope to find that the approximation improves as \( n \) increases. If we can in fact show that \( p_n(x) \to f(x) \) as \( n \to \infty \), then we will have shown that \( f(x) \) has the infinite series representation

\[
 f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(k)}(a)}{k!} (x-a)^k + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k.
\]  

(5.5)

**Definition 67** The infinite series in (5.5) is called the Taylor series associated with \( f \) at the base point \( a \). In case \( a = 0 \), the series \( \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \) is sometimes called the Maclaurin series associated with \( f \).

**Examples.** 1. Since \( \frac{d^k}{dx^k} e^x = e^x \) for each \( k \), and hence each derivative has the value 1 at \( x = 0 \) and \( e \) at \( x = 1 \), we have that the Maclaurin series of \( e^x \) is \( \sum_{k=0}^{\infty} \frac{1}{k!} x^k \) and the Taylor series at \( x = 1 \) is \( \sum_{k=0}^{\infty} \frac{e}{k!} (x-1)^k \).

  2. For \( f(x) = \ln(1+x) \) we have \( f'(x) = \frac{1}{1+x}, f''(x) = \frac{-1}{(1+x)^2} \) and more generally \( f^{(k)}(x) = (-1)^{k-1} \frac{(k-1)!}{(1+x)^k} \). Thus \( f^{(k)}(0) = (-1)^{k-1} (k-1)! \) This makes the Maclaurin series of \( \ln(1+x) \) the series \( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \). Note that this starts from \( k = 1 \), since \( \ln(1+0) = 0 \) and the term for \( k = 0 \) in the summation expression would require special handling.

5.1.1 Taylor Series by Substitution

Suppose that \( f \) has the Taylor series expansion \( \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \). I will abbreviate this by \( f \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \). (I don’t want to use an equals sign until we have proved that the series actually converges and to \( f(x) \).) Then the composite function \( g(x) = f(bx^m) \) has the Taylor series expansion \( \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (bx^m)^k = \sum_{k=0}^{\infty} \frac{b^k f^{(k)}(0)}{k!} x^{km} \) obtained by substituting \( bx^m \) for \( x \) in the Taylor series for \( f \). For instance, since we know \( e^x \sim \sum_{k=0}^{\infty} \frac{1}{k!} x^k \), we have \( e^{2x} \sim \sum_{k=0}^{\infty} \frac{1}{k!} (2x)^k = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k \) and \( e^{x^2} \sim \sum_{k=0}^{\infty} \frac{1}{k!} (x^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k} = 1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \ldots \). This can save a lot of complicated calculations with derivatives. For example, for \( g(x) = e^{x^2} \), we have

\[
 g'(x) = 2xe^{x^2}, g''(x) = (2 + 4x^2) e^{x^2}, g'''(x) = (12x + 8x^3) e^{x^2} 
\]
and so forth, where the polynomial multiplying $e^x$ keeps growing as you take successive derivatives in a way that it is hard to find a pattern for. (Note, however that if we do evaluate each of these three derivatives at $x = 0$ we do get the terms for $k = 1, 2, 3$ in the series for $e^x$ obtained by substitution.)

Substitution into Taylor series around points other than $a = 0$ is more complicated.

**EXERCISES**

Write the terms for $k \leq 4$ and the general term of the Taylor series for

1. (a) $e^{2x}$ around $a = 0$, (b) $e^{2x}$ around $a = 1$. Do these calculations both directly and by substitution and compare results.

2. (a) $\sin x$ around $a = 0$, (b) $\sin x$ around $a = \pi/2$.

3. (a) $\cos x$ around $a = 0$, (b) $\cos x$ around $a = \pi/2$.

4. (a) $\frac{1}{1 + x}$ around $a = 0$, (b) $\ln(1 + x)$ around $a = 0$.

5. (a) $3x^2 - 5x + 2$ around $a = 0$ (b) $3x^2 - 5x + 2$ around $a = 1$.

6. (a) $\sin(x^2)$ around $a = 0$, (a) $\sin(2x)$ around $a = 0$.

7. (a) $\cos(5x)$ around $a = 0$, (b) $\cos(x^3)$ around $a = 0$.

8. (a) $(1 + x)^{1/2}$ around $a = 0$.

   (b) $(1 + x)^p$ around $a = 0$ where $p$ is an arbitrary real number.

9. In one of Einstein’s papers on general relativity he had to solve for $\phi$ the equation

$$\sin \phi + b \left(1 + \cos \phi + \cos^2 \phi\right) = 0$$

where $b$ is a small positive constant.

(a) Solve this equation approximately by replacing each of the functions by its Taylor series around $a = 0$, ignoring all terms involving $\phi^2$ or higher powers of $\phi$ (this means you won’t actually have to calculate too many terms) and solving the resulting linear equation for $\phi$. (Your answer will involve $b$.)

(b) Solve approximately again, this time keeping $\phi^2$ terms and ignoring $\phi^3$ and higher terms. You will need to use the quadratic formula.

(c) Show that for very small $b$ one of the solutions from part (b) is approximately equal to the solution from part (a) by using the result of #8(a) (just the linear approximation) to approximate the square root term.

## 5.2 Power Series

We would like to show both that the Taylor series of the previous section converge, and that their sum gives the values of the function $f$ that we used to define the Taylor series. We will start by considering the convergence of such infinite series in a slightly more general context.
Definition 68 An infinite series of the form \( \sum_{k=0}^{\infty} c_k x^k \) or, more generally,
\[
\sum_{k=0}^{\infty} c_k (x - a)^k,
\]
is called a power series. Here the \( c_k \) are arbitrary real numbers. (No sign restrictions.)

We first investigate the nature of the set of values of \( x \) for which a power series converges. It is actually more convenient to consider absolute convergence.

Theorem 69 If the power series \( \sum_{k=0}^{\infty} c_k (x - a)^k \) converges absolutely for some \( x_0 \), then it also converges absolutely for each real number \( x \) with \( |x - a| \leq |x_0 - a| \). Thus the set of all \( x \) for which the series converges absolutely is an interval \( I \) centered at \( x = a \). (It is possible that \( I \) is all of \( \mathbb{R} \) or that \( I \) consists of the single point \( a \).)

Proof. This is immediate from the Comparison Theorem with \( a_k = |c_k| |x - a|^k \), \( b_k = |c_k| |x_0 - a|^k \).

Definition 70 If the interval \( I = (a - r, a + r) \), we call \( r \) the radius of convergence of the power series. Both \( r = 0 \) and \( r = \infty (I = \mathbb{R}) \) are possible. \( I \) is called the interval of convergence.

In most cases that arise in practice it is possible to identify the interval \( I \) using the Ratio Test. This works as follows.

Theorem 71 Let \( \sum_{k=0}^{\infty} c_k (x - a)^k \) be a power series. Suppose that \( r = \lim_{k \to \infty} \frac{|c_k|}{|c_{k+1}|} \) exists (possibly +\( \infty \)). Then \( r \) is the radius of convergence of the power series, that is, the series converges absolutely for \( |x - a| < r \) and diverges for \( |x - a| > r \). (If \( r = \infty \) the series converges absolutely for all \( x \).)

Proof. Use the Ratio Test. The ratio of successive terms of the series of absolute values is
\[
\frac{|c_{k+1}| (x - a)^{k+1}}{|c_k| (x - a)^k} = \frac{|c_{k+1}|}{|c_k|} |x - a| \to \frac{|x - a|}{r}
\]
as \( k \to \infty \). Thus the series converges absolutely when \( |x - a| < r \) and diverges when \( |x - a| > r \).

Remark. Note that if \( |x - a| = r \), then the ratios approach 1, which is the value where the Ratio Test gives no information. Thus the Ratio Test generally gives no information about whether the power series converges at the endpoints of the interval of convergence. In general the situation at the
endpoints is difficult: the series may either converge or not converge, and if it converges it may do so only conditionally rather than absolutely. In these notes we will mostly stick to studying what happens in the interior of the interval of convergence where we can count on absolute convergence and leave the situation at the endpoints to a more advanced course.

**Examples.** 1. The Taylor series for $e^x$ at $x = 0$ is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$. Thus here

$$\left| \frac{c_k}{c_{k+1}} \right| = \frac{(k+1)!}{k!} = k+1 \to \infty$$

so $r = \infty$, that is, the series converges for all $x$. Note however that we do not have the right to assume that it converges to $e^x$. We will consider this question in Section 5.4.

2. The Maclaurin series for $\ln (1 + x)$ is $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}x^k$, so

$$\left| \frac{c_k}{c_{k+1}} \right| = \frac{k+1}{k} \to 1$$

so $r = 1$, that is, the series converges absolutely for $-1 < x < 1$. Since the original function is not defined when $x = -1$, it is not surprising that $r = 1$.

**Remark.** Often the range of this result can be extended to include series where many of the coefficients $c_k$ are zero. For example, by direct calculation, $\cos x$ has Taylor series

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. $$

Theorem 71 does not apply because for each $k$, $c_{2k+1} = 0$. Thus the quotients are alternately 0 and $\infty$. However the substitution $u = x^2$ suggests that we look at the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k u^k}{(2k)!}. \quad (5.6)$$

For this series, $|c_k| = \frac{1}{(2k)!}$, so $\frac{c_k}{c_{k+1}} = (2k+1) (2k+2) \to \infty$. Thus this series converges for all real $u$. Now every partial sum of the original series is also a partial sum of the series (5.6). For instance the fourth and fifth partial sums of the original series (that is, terms through $x^3$ and $x^4$ respectively) for $x = -2$ are each equal to the third partial sum of (5.6) for $u = (-2)^2 = 4$. Thus the original series converges for all $x$ such that (5.6) converges for $x^2 = u$ that is, for all $x$.

A similar calculation leads to the conclusion that the Taylor series for $\sin x$,

$$x - \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

converges for all $x$. 

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EXERCISES. Find the radius of convergence and the interval of convergence. Don’t worry about convergence at the endpoints of the interval of convergence.

1. \( \sum_{k=0}^{\infty} k^2 x^k \)
2. \( \sum_{k=0}^{\infty} 3^k (x + 1)^k \)
3. \( \sum_{k=0}^{\infty} k^3 (5x - 3)^k \)
4. \( \sum_{k=1}^{\infty} (x - 2)^k \)
5. \( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \)
6. \( \sum_{k=0}^{\infty} \frac{k^2}{e^k} x^{2k} \)

7. The Maclaurin series for \((1 + x)^p\), where \(p\) is not necessarily an integer, is \( \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!} x^k \). (For \(p = 1/2\) this is \(1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{-1}{2}\right)x^2 \) + \(\frac{1}{2}\left(-\frac{3}{2}\right)x^3 + \ldots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\)) Find the radius of convergence of this series, called the binomial series.

Remark. It is important to realize that the calculations just before the exercises show only that the various Taylor series converge. They do not show that they converge to the functions we generated the series from. For instance, we now know that the Maclaurin series for \(\sin\) converges for all \(x\). But we do not know that it converges to \(\sin x\). We will have to establish this separately.

First, however, we show that functions defined by power series are “nice” functions, that is, they can be differentiated and integrated term by term just like polynomials.

**Theorem 72** Let \( f(x) = \sum_{k=0}^{\infty} c_k (x - a)^k \) be a function defined by a power series with positive radius of convergence \(r\). Then \(f\) is differentiable at each point of the open interval \((a - r, a + r)\) and for all \(x\) in this interval,

\[
f'(x) = \sum_{k=0}^{\infty} kc_k (x - a)^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} (x - a)^k.
\]

Thus the series obtained by term by term differentiation has the same radius of convergence \(r\) and converges to the derivative of \(f\).

**Proof.** To simplify notation a bit we will take \(a = 0\). Fix \(x_0\), \(|x_0| < r\). We will compute \(f'(x_0)\) as a limit of difference quotients using the fact that a polynomial \(p_n(x) = \sum_{k=0}^{n} c_k x^k\) has derivative \(p'_n(x) = \sum_{k=0}^{n} kc_k x^{k-1}\). The details will be a bit complicated, but the idea is to reduce the problem to this known property of polynomials by showing that the tails of all the series involved become as small as we like simply by choosing \(n\) large enough.

First we show that the series obtained by term by term differentiation converges. Convergence is trivial in case \(x_0 = 0\), so we may assume \(0 < x_0 < r\). Let \(x_1 = \frac{r + |x_0|}{2}\). Then \(|x_0| < x_1 < r\). Since the series \(\sum_{k=0}^{\infty} c_k x_1^k\) converges, we...
know \( c_k x^k \to 0 \). In particular, there is \( K \) such that \( |c_k| x^k < 1 \) for all \( k > K \). For any such \( k \) we have

\[
k |c_k x_0^{k-1}| = \frac{1}{|x_0|} (|c_k| x_1^k k) \left(\frac{|x_0|}{x_1}\right)^k < \frac{1}{|x_0|} k \left(\frac{|x_0|}{x_1}\right)^k = \frac{1}{|x_0|} k \left| \frac{x}{x_1} \right|^k
\]

where \( s = \frac{|x_0|}{x_1} < 1 \). Now the series \( \sum_{k=0}^{\infty} \frac{1}{|x_0|} k s^k \) converges by the Ratio Test, since \( \frac{(k+1)}{k} s \to s < 1 \). Thus \( \sum_{k=0}^{\infty} k c_k x_0^{k-1} \) converges absolutely by the Comparison Test. This shows that the series \( \sum_{k=0}^{\infty} k c_k x_0^{k-1} \) converges absolutely for \( |x| < r \). We denote the sum of this series by \( g(x) \).

It remains to show that \( f \) is differentiable and that \( f'(x) = g(x) \) for each \( x \) with \( |x| < r \). Consider the difference quotient

\[
\frac{f(x) - f(x_0)}{x - x_0} = \frac{1}{x - x_0} \sum_{k=0}^{\infty} c_k (x^k - x_0^k)
\]

where we may assume that \( x \) is close enough to \( x_0 \) so that \( |x| < x_1 \) and let \( \epsilon > 0 \) be given. We will break the series up into two parts as follows. Since \( \sum_{k=0}^{\infty} k c_k x_1^{k-1} \) converges absolutely, there is \( N \) so that \( n > N \) implies that

\[
\left| \sum_{k=0}^{n} k |c_k| x_1^{k-1} - \sum_{k=0}^{N} k |c_k| x_1^{k-1} \right| < \frac{\epsilon}{3},
\]

that is,

\[
\sum_{k=N+1}^{\infty} k |c_k| x_1^{k-1} < \frac{\epsilon}{3}.
\]

Now \( |x^k - x_0^k| = |x - x_0| |x^{k-1} + x^{k-2} x_0 + x^{k-3} x_0^2 + \ldots + x_0^{k-1}| \leq |x - x_0| (k x_1^{k-1}) \) since there are \( k \) terms in the sum and each term satisfies \( |x^j x_0^{k-j-1}| < x_1^{k-1} \).

Thus

\[
\left| \sum_{k=N+1}^{\infty} c_k (x^k - x_0^k) \right| \leq \sum_{k=N+1}^{\infty} |c_k| |x^k - x_0^k| \leq |x - x_0| \sum_{k=N+1}^{\infty} k |c_k| x_1^{k-1} < |x - x_0| \frac{\epsilon}{3}.
\]

Also, by the Comparison Theorem, \( \sum_{k=N+1}^{\infty} k |c_k| x_0^{k-1} < \frac{\epsilon}{3} \). Thus

\[
\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| = \left| \frac{1}{x - x_0} \sum_{k=0}^{\infty} c_k (x^k - x_0^k) - \sum_{k=0}^{N} k c_k x_0^{k-1} \right| \leq
\]

\[
\leq \left| \frac{1}{x - x_0} \sum_{k=0}^{n} c_k (x^k - x_0^k) - \sum_{k=0}^{N} k c_k x_0^{k-1} \right| +
\]

\[
+ \frac{1}{|x - x_0|} \left| \sum_{k=N+1}^{\infty} c_k (x^k - x_0^k) \right| + \left| \sum_{k=N+1}^{\infty} k c_k x_0^{k-1} \right| \leq
\]

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\[ \leq \left| \frac{1}{x-x_0} \sum_{k=0}^{n} c_k (x^k - x_0^k) - \sum_{k=0}^{n} kc_k x_0^{k-1} \right| + \frac{\epsilon}{3} + \frac{\epsilon}{3}. \]

The first term here is just
\[ \frac{p_n (x) - p_n (x_0)}{x - x_0} - p_n' (x_0) \]
for the polynomial \( p_n (x) = \sum_{k=0}^{n} c_k x^k \). We know that there is \( \delta > 0 \) such that
\[ \left| \frac{p_n (x) - p_n (x_0)}{x - x_0} - p_n' (x_0) \right| < \frac{\epsilon}{3} \]
whenever \( 0 < |x - x_0| < \delta \). For such \( x \),
\[ \left| \frac{f (x) - f (x_0)}{x - x_0} - g (x_0) \right| < \epsilon. \]

Since for every \( \epsilon > 0 \) we can find \( \delta > 0 \) such that this is true whenever \( 0 < |x - x_0| < \delta \), the function \( f \) is differentiable at \( x_0 \) and \( f' (x_0) = g (x_0) \).

This theorem has a number of important consequences. The first one shows that Taylor series are the only interesting power series.

**Theorem 73** Let \( f (x) = \sum_{k=0}^{\infty} c_k (x - a)^k \) have positive radius of convergence \( r \). Then \( f \) has derivatives of all orders at \( x = a \) and for each \( n \),
\[ f^{(n)} (a) = n!c_n. \]

**Proof.** It is clear that \( f (a) = f ^{(0)} (a) = c_0 \), since all other terms in the series vanish when \( x = a \). From the theorem on term by term differentiation, \( f' \) exists for \( |x - a| < r \) and
\[ f' (x) = \sum_{k=0}^{\infty} (k + 1) c_{k+1} (x - a)^k. \]

In particular, \( f' (a) = c_1 \) since again all other terms of the series vanish for \( x = a \).

Now we can apply the theorem on term by term differentiation again to see that \( f'' \) exists and to obtain a series representation for \( f'' \). Once again, \( f'' (a) \) is simply the constant term in the sum, in this case \( f'' (a) = 2c_2 \). Continuing in this way (strictly speaking, using mathematical induction) the theorem follows.

**Corollary 74** The power series representation of a function \( f \) on an interval \( I \) is unique, that is, if
\[ f (x) = \sum_{k=0}^{\infty} c_k (x - a)^k \quad \text{and} \quad f (x) = \sum_{k=0}^{\infty} d_k (x - a)^k \]
for all \( x \) such that \( |x - a| < r \), then \( c_k = d_k \) for each \( k = 0, 1, 2, \ldots \).
Proof. By the previous theorem,

\[ c_k = \frac{f^{(k)}(a)}{k!} = d_k. \]

**Remark.** One might think that the last theorem and its corollary would show that every infinitely differentiable function is the limit of its Taylor series. In fact a more careful reading will confirm that it shows only that if a function is equal to the sum of a power series, then that series must be its Taylor series. To demonstrate that the "if" cannot be ignored, we give an example of an infinitely differentiable function that is **not** the sum of its Taylor series.

**Example.** Let 

\[ f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases} \]

Note that \( f(x) > 0 \) for all \( x \neq 0 \). I claim that for each positive integer \( k \), \( f^{(k)}(0) \) exists and equals 0. If so, then the Taylor series for this function is the zero series and so convergent for all \( x \) but not equal to \( f(x) \) except at \( x = 0 \).

We have

\[ f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1}{x} e^{-1/x^2} = \lim_{u \to \infty} u e^{-u^2} = 0 \]

where \( u = \frac{1}{x} \) and the last equality can be established in various different ways. (l'Hopital’s Rule is one, but a simple examination of this function using calculus is another.) By the way, the difference quotient approaches zero very quickly. My calculator says that its value for \( x = 0.1 \) is about \( 4 \times 10^{-43} \). This establishes the claim for \( k = 1 \). For the next stage, we know that for \( x \neq 0 \), \( f'(x) = \frac{2}{x} e^{-1/x^2} \). We can show \( f''(0) = 0 \) by a similar investigation of the difference quotient. In general we can proceed by mathematical induction. In effect what must be established is that for any polynomial \( p \),

\[ \lim_{u \to \infty} p(u) e^{-u^2} = 0 \]

or equivalently, for each \( n \), \( u^n e^{-u^2} \to 0 \) as \( u \to \infty \).

Finally, we use the theorem on term by term differentiation to get the corresponding result for term by term anti-differentiation.

**Theorem 75** Let \( f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k \) have positive radius of convergence \( r \). Then

\[ F(x) = \int_a^x f(t) \, dt \]

has the series representation

\[ F(x) = \sum_{k=0}^{\infty} \frac{c_k}{k+1} (x-a)^{k+1} \quad (5.7) \]

in the interval \( |x-a| < r \).
Proof. That the series in (5.7) converges absolutely for each \( x_0 \) with \( |x_0 - a| < r \) to some function \( G(x) \) can be seen in the same way as in the previous theorem. If \( x_1 - a = \frac{|x_0 - a| + r}{2} \), then \( |c_k| (x_1 - a)^k \to 0 \) as \( k \to \infty \) and convergence follows by comparison with a convergent geometric series.

It then follows from the previous theorem that \( G \) is differentiable for \( |x - a| < r \) and that \( G'(x) \) is given by term by term differentiation, that is

\[
G'(x) = \sum_{k=0}^{\infty} c_k (x - a)^k = f(x).
\]

Thus \( G \) is an antiderivative of \( f \). Finally, it is clear that \( G(a) = 0 = \int_{a}^{a} f(t) \, dt = F(a) \). Thus \( G = F \) and the theorem is proved.

EXERCISES.

1. How can we tell at a glance from the Maclaurin series for a function \( g \) whether \( g \) has a critical point at the origin?

2. (a) Recall that the Second Derivative Test says that the sign of \( g''(0) \) tells you whether a critical point of \( g \) at \( x = 0 \) is a local maximum or a local minimum. Which sign goes with which outcome? (Draw a picture of \( g \) and remember that \( g'' > 0 \) means that \( g' \) is increasing if you don’t remember the statement of the test.)

(b) Restate the Second Derivative Test in this form: If \( g \) has a critical point at \( x = 0 \) and if the coefficient of \( x^2 \) in the Maclaurin series for \( g \) is \_, then \( g \) has a local maximum at \( 0 \); if the coefficient of \( x^2 \) in the Maclaurin series for \( g \) is \_, then \( g \) has a local minimum at \( 0 \). Explain why this is reasonable in terms of the graph of the polynomial \( p_2(x) = a_0 + a_1 x + a_2 x^2 \) consisting of the first three terms (not all different from zero) of the Maclaurin series for \( g \).

(c) The Second Derivative Test states that if \( g''(0) = 0 \), then no conclusion can be drawn about the nature of the critical point at \( 0 \). What does the Maclaurin series look like in this case? What feature of the Maclaurin series can be used in this situation to determine whether \( 0 \) is a local maximum, a local minimum, or neither?

3. Assuming that the Maclaurin series for \( f(x) = x^2 e^{x^2} \) is \( x^2 + x^4 + \frac{1}{2!} x^6 + \frac{1}{3!} x^8 + ... \), find

\[ \frac{d}{dx} \left( x^2 e^{x^2} \right) \bigg|_{x=0} \quad \text{and} \quad \frac{d^4}{dx^4} \left( x^2 e^{x^2} \right) \bigg|_{x=0}. \]

4. Let \( f(x) = \sum_{k=0}^{\infty} (k+1)(x-3)^k = 1 + 2(x-3) + 3(x-3)^3 + ... \). By inspection find \( f(3), f'(3), f''(3) \).
5.3 Shortcuts for Computing Taylor Series

Here are some shortcuts for computing Taylor series. They will be justified by the results of the next section.

1. **Substitution**: If \( f(x) = \sum_{k=0}^{\infty} c_k x^k \) for \( |x| < r \), then for any \( j \) and ny real number \( c \), \( f(\pm (cx)^j) = \sum_{k=0}^{\infty} c_k \left( \pm (cx)^j \right)^k = \sum_{k=0}^{\infty} (-1)^k c_k c^j x^{jk} \) when \( |cx^j| < r \), that is, \( |x| < r^{1/j} |c| \).

   For instance,
   \[
e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} (x^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}.
   \]

   Similarly,
   \[
   \cos 3x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (3x)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 3^{2k} x^{2k}.
   \]

2. **Geometric series**: We know that for \( |x| < 1 \), \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \). By Theorem 73, this must be the Taylor series for \( \frac{1}{1-x} \). Now, using the substitution principle, we have for instance

   \[
   \frac{1}{1 + 4x^2} = \frac{1}{1 - (-4x^2)} = \sum_{k=0}^{\infty} (-1)^k 2^{2k} x^{2k}.
   \]

   Note that the radius of convergence is \( r = \frac{1}{2} \) compared to the radius of convergence 1 for \( \frac{1}{1-x} \), as predicted by the previous part.

   A more sophisticated example of the use of substitution and geometric series: to find the Taylor series for \( \frac{1}{x} \) around \( a = 2 \), write

   \[
   \frac{1}{x} = \frac{1}{2 - (2 - x)} = \frac{1}{2} \frac{1}{1 - (2 - x)/2} = \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{2 - x}{2} \right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} (x - 2)^k.
   \]

3. **Integration**: For instance, we know \( \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - ... \), so

   \[
   \arctan x = \int_0^x \frac{1}{1 + t^2} dt = x - \frac{x^3}{3} + \frac{x^5}{5} - ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.
   \]
4. **Differentiation**: Since \( \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2} \),

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \left( \sum_{k=0}^{\infty} x^k \right) = 1 + 2x + 3x^2 + \ldots = \sum_{k=0}^{\infty} (k+1) x^k.
\]

5. **Multiplication**: A simple case is

\[
x^2 \cos x = x^2 \left( 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots \right) = x^2 - \frac{1}{2!} x^4 + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+2}.
\]

More advanced:

\[
\sin x \cos x = \left( x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \ldots \right) \left( 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots \right) = \\
x - \left( \frac{1}{2!} + \frac{1}{3!} \right) x^3 + \left( \frac{1}{4!} + \frac{1}{2!} \frac{1}{3!} + \frac{1}{5!} \right) x^5 - \ldots = x - \frac{2}{3} x^3 + \frac{2}{15} x^5 - \ldots
\]

For comparison,

\[
\sin x \cos x = \frac{1}{2} \sin 2x = \frac{1}{2} \left( 2x - \frac{1}{3!} (2x)^3 + \frac{1}{5!} (2x)^5 - \ldots \right) = x - \frac{2}{3} x^3 + \frac{2}{15} x^5 - \ldots
\]

6. **Division**: to determine the Taylor series \( \sum_{k=0}^{\infty} c_k x^k \) for \( \tan x = \frac{\sin x}{\cos x} \),

\[
c_0 + c_1 x + c_2 x^2 + \ldots = \frac{x - \frac{1}{3!} x^3 + \ldots}{1 - \frac{1}{2!} x^2 + \ldots}
\]

or

\[
x - \frac{1}{3!} x^3 + \ldots = (c_0 + c_1 x + c_2 x^2 + \ldots) \left( 1 - \frac{1}{2!} x^2 + \ldots \right) = \\
c_0 + c_1 x + (c_2 - \frac{1}{2!} c_0) x^2 + (c_3 - \frac{1}{2!} c_1) x^3 + \ldots
\]

so, matching coefficients between the left and right sides of the equation,

\[
c_0 = 0, c_1 = 1, c_2 = 0, c_3 = \frac{1}{3}, \ldots
\]

5.3.1 Some Useful Taylor Series.

1. \( e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \)

2. \( \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \)
3. \( \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!}x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \)

4. \( (1 + x)^p = \sum_{k=0}^{\infty} \frac{p(p - 1)(p - 2)\ldots(p - k + 1)}{k!}x^k = 1 + px + \frac{p(p - 1)}{2!}x^2 + \frac{p(p - 1)(p - 2)}{3!}x^3 + \ldots \)

5. \( \frac{a}{b - x} = \frac{a}{b} \frac{1}{1 - x/b} = \frac{a}{b} \sum_{k=0}^{\infty} \frac{1}{b^k} x^k; \quad \frac{a}{b + x} = \frac{a}{b} \frac{1}{1 + (-x/b)} = \frac{a}{b} \sum_{k=0}^{\infty} \frac{(-1)^k}{b^k} x^k \)

**EXERCISES.**

1. Show that if you integrate the Maclaurin series for cosine term by term, you get the Maclaurin series for sine.

2. Find the Maclaurin series for \( \frac{1}{(1 + x)^2} \) by differentiating the Maclaurin series for \( \frac{1}{1 + x} \) term by term. Verify that this works by also computing the first few terms directly from the definition of Taylor series (by computing successive derivatives at \( x = 0 \)).

3. Expand \( \frac{1}{1 + x^2} \) as a geometric series. Integrate term by term to get the Maclaurin series for \( \arctan x \).

4. Integrate the geometric series with sum \( \frac{1}{1 + x} \) term by term to get the Maclaurin series for \( \ln (1 + x) \).

5. Find a closed form (that is, a formula) for the sum of the series \( 1 + 2x + 3x^2 + \ldots \) by noting that you can integrate this term by term to get a geometric series and that this series is the derivative of that one.

6. Use the result of #5 to find a closed form sum for \( 1 + x + 2x^2 + 3x^3 + \ldots \).

7. If \( f(x) = \sum_{k=0}^{\infty} (k+1)(x-3)^k = 1 + 2(x-3) + 3(x-3)^3 + \ldots \), find the Taylor series about \( a = 3 \) for \( F(x) = \int_3^x f(t) \, dt \). Use the fact that it is a geometric series to find a simple formula for \( F \). Differentiate to find a simple formula for \( f \). Compute the first few derivatives at \( a = 3 \) to verify that your formula agrees with the original definition of \( f \) in terms of its Taylor series.

8. Find the Maclaurin series for \( \frac{\sin x}{x} \). Explain how this tells you what \( \lim_{x \to 0} \frac{\sin x}{x} \) is.

9. Use division to determine the first two nonzero terms of the Maclaurin series for \( \frac{1 - \frac{\sin x}{x}}{1 - \cos x} \). Express the coefficients as fractions rather than decimals. Use the series to determine \( \lim_{x \to 0} \frac{1 - \frac{\sin x}{x}}{1 - \cos x} \). Compare values for \( \frac{1 - \frac{\sin x}{x}}{1 - \cos x} \) for...
$x = \pm 1, \pm 0.5, \pm 0.1, \pm 0.01$ to values given by the first two terms of the series. Keep enough decimal places in each case to be able to distinguish between the correct value and the series approximation.

## 5.4 Convergence of Taylor Series

It follows from the example of $f(x) = e^{-1/x^2}$ discussed at the end of section 5.2 that not every differentiable function is the sum of its Taylor series. We have, therefore, something to investigate. We will establish a general estimate of the difference between a function and the n-th partial sum of its Taylor series at a point $a$. This can be used to show that a function $f$ is the sum of its Taylor series in some interval containing $a$ provided we are able to estimate $\max |f^{(k)}(x)|$ suitably in this interval. In practice, this is generally possible.

**Theorem 76** Let $f$ be $n+1$ times differentiable in an interval $I = (a - r, a + r)$. Then for each $x$ with $|x - a| < r$,

$$|f(x) - \left[ f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right]| \leq \max_{t} |f^{(n+1)}(t)| \frac{|x-a|^{n+1}}{(n+1)!}.$$ 

Here the maximum is over the interval with endpoints $a$ and $x$.

**Proof.** Note that the estimate for the error is essentially just the next term in the Taylor series, except that instead of $f^{(n+1)}(a)$ we must use $\max_{t} |f^{(n+1)}(t)|$.

By the Fundamental Theorem of Calculus,

$$|f(x) - f(a)| = \left| \int_{a}^{x} f'(t) \, dt \right| \leq \int_{a}^{x} |f'(t)| \, dt \leq \max_{t} |f'(t)| |x-a|.$$  \hspace{1cm} (5.8)

This is just the conclusion of the theorem for $n = 0$. Continuing with the first integral in (5.8), integrating by parts with $u = f'(t)$, $dv = dt$, $v = t - x$ (remember $x$ is a constant in this integral) gives

$$\int_{a}^{x} f'(t) \, dt = (t - x) f'(t)|_{a}^{x} - \int_{a}^{x} f''(t) \, dt =$$

$$= f'(a)(x-a) - \int_{a}^{x} (t - x) f''(t) \, dt.$$  \hspace{1cm} (5.9)

Thus

$$|f(x) - f(a) - f'(a)(x-a)| = \left| \int_{a}^{x} (t - x) f''(t) \, dt \right| \leq$$

$$\leq \max_{t} |f''(t)| \int_{a}^{x} (x-t) \, dt = \max_{t} |f''(t)| \left( \frac{(x-t)^2}{2} \right)_{a}^{x}$$

$$= \frac{\max_{t} |f''(t)|}{2} |x-a|^2.$$
Continuing in this way, we can establish by induction that for each $n$, 

\[ f(x) - \left[ f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n \right] = \frac{(-1)^n}{n!} \int_a^x f^{(n+1)}(t) (t - x)^n \, dt. \]

(The way to think of this formula is that each side represents a function $g(x)$ such that $g^{(k)}(a) = 0$ for each $k = 0, 1, \ldots, n$ and $g^{(k)}(a) = f^{(k)}(a)$ for each $k > n$.) Then as in the case $n = 1$, 

\[
\left| \frac{1}{n!} \int_a^x f^{(n+1)}(t) (t - x)^n \, dt \right| \leq \frac{\max_t |f^{(n+1)}(t)|}{n!} \left| \int_a^x (x - t)^n \, dt \right| = \frac{\max_t |f^{(n+1)}(t)|}{(n + 1)!} |x - a|^{n+1}
\]

and the theorem is proved.

**Examples.**

1. For $f(x) = \cos x$ we have $f \sim \sum_{k=0}^{\infty} a_k x^k$ where $a_k = \begin{cases} \frac{(-1)^{k/2}}{k!}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$. Also $|f^{(n+1)}(t)| \leq 1$ for each $n$ and each $t$. Thus for any $x$ and any $n$,

\[
\left| f(x) - \sum_{k=0}^{n} a_k x^k \right| \leq \frac{1}{(n + 1)!} |x|^{n+1} = A_{n+1}.
\]

As $n \to \infty$, $\frac{A_{n+1}}{A_n} = \frac{|x|}{(n+1)} \to 0$, so $A_{n+1} \to 0$ as $n \to \infty$. Thus for all $x$,

\[
\cos x = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k)!} x^{2k}.
\]

By a similar argument, for all $x$,

\[
\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}.
\]

2. For $f(x) = e^x$ we have that $\max_{t \leq |x|} |e^t| = e^{|x|}$, so

\[
\left| f(x) - \sum_{k=0}^{n} \frac{1}{k!} x^k \right| \leq \frac{e^{|x|}}{(n + 1)!} |x|^{n+1} \to 0 \text{ as } n \to \infty.
\]

**EXERCISES.**

Use the Remainder Formula given in Theorem 76 for the estimates in these problems.
1. If we use \( p(x) = 1 + x + x^2/2 + x^3/6 + x^4/24 \) as an estimate for \( e^x \), how close can we expect the estimate to be for \( x = 1 \)? For \( x = .1 \)? For \( x = 10 \)? Then compute \( p(x) \) for \( x = 1, .1, 10 \) to see how close each really is to \( e^x \).

2. If we use \( p(x) = x - x^3/3! + ... - x^{15}/15! \) as an estimate for \( \sin x \), how close can we expect the estimate to be for \( x = \pi/2 \)? Within what interval centered at the origin can we be certain that \( |\sin x - p(x)| < .01 \)?

3. Find the Taylor series for \( \sin x \) at \( a = \pi/2 \) through the \( (x - \pi/2)^3 \) term. In what interval around \( \pi/2 \) can we be certain that this polynomial is within .01 of \( \sin x \)? Sketch the graph of \( \sin x \) and this Taylor polynomial for \( 0 \leq x \leq \pi \).

5.5 An Application: Power Series Solutions of Differential Equations

Recall that a **differential equation** is just an equation involving one or more derivatives of a function. Usually the function is, at least initially, unknown, and the problem is to deduce the function, or at least as much information about it as possible, from the equation relating it to one or more of its derivatives. For example, the simplest, and possibly most important differential equation is

\[
\frac{dy}{dt} = y.
\]

A **solution** to this differential equation would be any function that is equal to its own derivative. We know that such functions are precisely those of the form \( y = Ce^t \), where \( C \) is a constant. Notice that if we specify the value of \( y \) at one point, say \( y(0) = 3 \), then the equation together with this one initial condition determine a unique solution (in this case \( y = 3e^t \)).

The **order** of a differential equation is the highest derivative that appears in the equation. Thus the equation above is a **first order** equation. An example of a second order differential equation is

\[
\frac{d^2y}{dt^2} + 4y = 0.
\]

It can be verified that \( \sin 2t \) and \( \cos 2t \) are both solutions of this differential equation. Moreover, for any constants \( c_1 \) and \( c_2 \), so is \( c_1 \cos 2t + c_2 \sin 2t \). It turns out that all solutions of this equation are of this form for some choice of the constants \( c_1 \) and \( c_2 \). (It is typical that specifying all solutions of a first order equation involves one arbitrary constant, and that specifying all solutions of a second order equation involves two arbitrary constants.) One could determine a unique solution of this equation by specifying two extra conditions, say \( y(0) = 1 \), \( y'(0) = -2 \). It is easy to verify that \( \cos 2t - \sin 2t \), that is, \( c_1 = 1, c_2 = -1 \) is the only function in this family satisfying both conditions.
Each of these equations can be solved by methods studied in an introductory differential equations course. Here we will see how to solve them by power series methods. The advantage of the power series method is that it can be used in situations where the methods leading directly to closed form solutions do not apply.

**Example 1.** Solve $y' = y$. The general idea is to assume $y$ is given by a power series, $y = \sum_{k=0}^{\infty} c_k t^k$, where of course we have no information initially about the coefficients $c_k$, and to get equations from which to determine the $c_k$ by plugging the series into the equation and equating the coefficients of equal powers of $t$. Specifically, if

$$y = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n + \ldots,$$

then we know that

$$y' = c_1 + 2c_2 t + 3c_3 t^2 + \ldots + (n+1)c_{n+1} t^n + \ldots.$$

We also know that the Taylor series representation of a function is unique, so that if $y' = y$, then the two series must be the same, that is, for each $n$, the coefficient of $t^n$ must be the same. Comparing the two series, this gives the sequence of equations

$$c_0 = c_1 \,(\text{comparing constant terms}),$$

$$c_1 = 2c_2 \,(\text{comparing coefficients of } t),$$

$$c_2 = 3c_3,$$

and in general,

$$c_n = (n+1)c_{n+1}.$$

We can use these equations to determine all the $c_n$’s in terms of $c_0$. We have

$$c_1 = c_0, \quad c_2 = \frac{c_1}{2} = \frac{c_0}{2}, \quad c_3 = \frac{c_2}{3} = \frac{c_0}{3 \cdot 2} = \frac{c_0}{3!}, \ldots.$$ 

The idea is to work a few out, and then try to see a general pattern. In this case is looks like the pattern might be

$$c_n = \frac{c_0}{n!}.$$ 

We verify this by mathematical induction. Suppose that this is true, and see whether the next term comes out correctly. We know from above that

$$c_{n+1} = \frac{c_n}{n+1} = \frac{1}{n+1} \cdot \frac{c_0}{n!} = \frac{c_0}{(n+1)!}.$$ 

Thus the pattern is verified, and we have that the general solution of the equation is

$$y = \sum_{k=0}^{\infty} \frac{c_0}{k!} t^k = c_0 \sum_{k=0}^{\infty} \frac{1}{k!} t^k.$$ 

**AN APPLICATION: POWER SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS**
Of course we recognise the series as representing \( e^t \), so we can write finally \( y = c_0 e^t \), just as before.

**Example 2.** Solve \( y'' + 4y = 0 \). This time, remembering the solution above, we will expect to have two undetermined constants. With \( y = c_0 + c_1 t + c_2 t^2 + \ldots + c_k t^k + \ldots \), we will have

\[
y'' = 2 \cdot 1c_2 + 3 \cdot 2c_3 t + 4 \cdot 3c_4 t^2 + \ldots + (k + 2) (k + 1) c_{k+2} t^k + \ldots
\]

Notice that this time when we equate coefficients of like powers of \( t \) we get one relationship among the coefficients with even indices and an entirely separate one among coefficients with odd indices. Specifically, from the even indices

\[
2c_2 = -4c_0, \quad 12c_4 = -4c_2, \quad 30c_6 = -4c_4, \ldots, \quad (2k + 2) (2k + 1) c_{2k+2} = -4c_{2k}, \ldots
\]

and from the odd indices

\[
6c_3 = -4c_1, \quad 20c_5 = -4c_3, \ldots, \quad (2k + 1) (2k) c_{2k+1} = -4c_{2k-1}, \ldots
\]

Solving successively for the coefficients with even indices,

\[
c_2 = \frac{-4c_0}{2}, \quad c_4 = \frac{-4c_2}{4 \cdot 3} = \frac{(-4)^2 c_0}{4 \cdot 3 \cdot 2}, \quad c_6 = \frac{-4c_4}{6 \cdot 5} = \frac{(-4)^3 c_0}{6!}.
\]

This suggests the general pattern

\[
c_{2k} = \frac{(-4)^k c_0}{(2k)!}.
\]

(Note that it was easier to spot the pattern by not simplifying the expressions for the first few coefficients.) To verify it, we look at the next one:

\[
c_{2k+2} = \frac{-4c_{2k}}{(2k + 2) (2k + 1)} = \frac{(-4)^k (-4) c_0}{(2k + 2) (2k + 1) (2k)!} = \frac{(-4)^k c_0}{(2k + 2)!} = \frac{(-4)^{k+1} c_0}{(2k + 2)!}.
\]

This part of the solution is then

\[
\sum_{k=0}^{\infty} \frac{(-4)^k c_0}{(2k)!} t^{2k} = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2t)^{2k} = c_0 \cos 2t.
\]

(Notice that it can be helpful to be thoroughly familiar with the Taylor series of some standard functions in order to recognise these series as they emerge.)

Similarly, looking at the terms with odd indices,

\[
c_3 = \frac{-4c_1}{3 \cdot 2}, \quad c_5 = \frac{-4c_3}{5 \cdot 4} = \frac{(-4)^2 c_1}{5!}, \ldots, \quad c_{2k+1} = \frac{(-4)^k c_{2k-1}}{(2k + 1) (2k)} = \frac{(-4)^k c_1}{(2k + 1)!}
\]

so that this part of the solution is

\[
\sum_{k=0}^{\infty} \frac{(-4)^k c_1}{(2k + 1)!} (2k+1) t^{2k+1} = \frac{c_1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} (2t)^{2k+1} = \frac{c_1}{2} \sin 2t.
\]
Thus the complete power series solution to the differential equation is

\[ c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2t)^{2k} + \frac{c_1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2t)^{2k+1} = c_0 \cos 2t + \frac{c_1}{2} \sin 2t. \]

The perhaps slightly unexpected coefficient of \( \sin 2t \) may be explained as follows. When we start with \( y = c_0 + c_1 t + c_2 t^2 + \ldots \) we are implicitly specifying \( y(0) = c_0, y'(0) = c_1 \) since this is what you get from the original series. So the coefficient of \( \sin 2t \) must come out so that \( \frac{d}{dt} \left( c_0 \cos 2t + \frac{c_1}{2} \sin 2t \right) = -2c_0 \sin 2t + c_1 \cos 2t \) evaluates to \( c_1 \) at \( t = 0 \).

**Exercises.** Use power series to find the general solution of each equation, and the particular solution satisfying the given initial condition. Find a closed form expression for the solution using well-known functions whose Taylor series is the series solution you have found.

1. \( y' - 2y = 0, y(0) = 1 \).
2. \( y' + y = 0, y(0) = 2 \).
3. \( y' - 2ty = 0, y(0) = 1 \).
4. \( y' + y = 1, y(0) = 1 \). Also \( y(0) = 0 \).
5. \( y'' + y = 0, y(0) = 0, y'(0) = 1 \).
6. \( y'' - y = 0, y(0) = 0, y'(0) = 1 \).
7. Show that the only solutions of \( ty'' + y' + ty = 0 \) that can be represented by a Maclaurin series are the multiples of the solution with \( c_0 = 1 \). This solution is called the **Bessel function of order zero**. Differential equations of this type (and their solutions, called Bessel functions) occur in the solution of certain important partial differential equations of mathematical physics. The series does not have any simple closed form representation, and the Maclaurin series representation is generally taken as the definition of the Bessel function. Because the functions are important, extensive tables of their values have been compiled, just as extensive tables of trigonometric functions have been compiled. Just as you can now find values of trig functions directly from your calculator, you can find values of Bessel functions directly from some computer algebra systems, such as Mathematica.
8. Find the general solution to \( y'' - ty = 0 \). The solutions to this equation are called **Airy functions**. They also occur in mathematical physics. As with Bessel functions, they have no simple expression in terms of more familiar functions, so we have to be content with the infinite series representation.
9. Use series to find a polynomial solution to \( ty'' + (1 - t) y' + 3y = 0 \). (Hint: With the right choice of first constants, \( c_k = 0 \) for \( k \geq 4 \).) This polynomial is a **Laguerre polynomial**.
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