Brooks' Theorem

Recall that the *greedy algorithm* shows that $\chi(G) \leq \Delta(G)+1$ for any graph G. Brooks' Theorem extends this assertion.

Theorem 1. *If G is connected, $\chi(G) \leq \Delta(G)$ unless G is complete or an odd cycle.*

Proof. We may assume $\Delta = \Delta(G) \geq 3$, since the result is easy otherwise. Our proof proceeds by induction on Δ, and, for each Δ, we will use induction on n. The induction starts at $n = \Delta+1$, and the theorem is true in this case, since if $|G| = n+1$ and $G \neq K_{n+1}$ we can colour G with Δ colours by using the same colour for some two non-adjacent vertices. Therefore, suppose $n \geq \Delta + 2$.

Case 1. There is a vertex v such that $G-v$ is disconnected. Let the components of $G-v$ be $C_1, \ldots, C_t$. Consider the graphs induced by G on the vertex sets $C_1 \cup \{v\}, \ldots, C_t \cup \{v\}$. We may Δ-colour each of these graphs by induction (if one of the graphs is complete or an odd cycle, *its* maximum degree must be strictly less than Δ). Switching colours within some of these colourings if necessary, we may assume that v gets colour 1 in all t colourings, which we can therefore combine to get a Δ-colouring of G.

Case 2. $G - v$ is connected for all v, but there are two non-adjacent vertices v and w such that $G - v - w$ is disconnected. You will understand the following argument better if you draw some figures to illustrate it.

Let A be a component of $G - v - w$, and let $B = V(G) \setminus (V(A) \cup \{v, w\})$. If there are no edges from v to A, then $G - w$ is disconnected, which we are assuming is not the case. Therefore, there is at least one edge from v to A. Similarly, there is at least one edge from w to A, at least one edge from v to B, and at least one edge from w to B.

Write G_1 for the graph obtained from G by deleting B, and G_2 for the graph obtained from G by deleting A. It is tempting at this point to Δ-colour G_1 and G_2 by induction and then combine the colourings, but it may not be possible to combine the colourings (to see why, consider the case when G is an odd cycle). Instead, we note that, from the above observations, v and w have degree at most $\Delta-1$ in both G_1 and G_2, so that we may Δ-colour $G_3 = G_1 + vw$ and $G_4 = G_2 + vw$ by induction, unless one of them is complete (if either of them is an odd cycle, we can Δ-colour it since $\Delta > 2$). Such colourings, if they exist, *can* be combined because v and w will be forced to have *different* colours in both of them: we can then switch colours if necessary to ensure that v and w are coloured 1 and 2 respectively in both colourings.

If G_3 is a clique on $\Delta + 1$ vertices, then each of v and w must have degree 1 in G_2 (since both have degree Δ in G_3 and $\Delta - 1$ in G_1). In G_2, we can combine v and w into a single vertex, obtaining a graph G_5, which can be Δ-coloured by induction. Therefore, there are Δ-colourings of both G_1 and G_2 in which both v and w get the same colour. These colourings can be combined to provide a Δ-colouring of G.

Case 3. $G - v - w$ is connected for every pair of non-adjacent vertices v and w. Select a vertex u of maximum degree Δ. Since $G \neq K_n$, some pair of neighbours v and w of u are not adjacent. We define $v_1 = v, v_2 = w, v_n = u$ and, working backwards from v_{n-1} to v_3, we ensure that each v_i has some neighbour among $\{v_{i+1}, \ldots, v_n\}$: this is possible since $G - v - w$ is connected. Running the greedy algorithm with this ordering of the vertices, we see that $v_1 = v$ and $v_2 = w$ both get colour 1, and also that we never need to use colour $\Delta + 1$ on $v_3, \ldots, v_{n-1}$, since each such v_i has only at most $\Delta - 1$ neighbours among the already coloured vertices. Finally, when we come to colour v_n, two of its Δ neighbours have received the same colour (1), so that one of the colours $1, \ldots, \Delta$ is available to colour v_n itself. This completes the induction step.

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