Secrecy coverage in two dimensions

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Abstract

Let $P$ and $P'$ be independent Poisson processes, of intensities 1 and $\lambda$ respectively, in $\mathbb{R}^2$. Place an open disc $D(p, r_p)$ of radius $r_p$ around each point $p \in P$, where $r_p$ is maximal so that $D(p, r_p) \cap P' = \emptyset$. We thus obtain a random set $A_\lambda \subset \mathbb{R}^2$ which is the union of discs centered at the points of $P$. Now let $B_n \subset \mathbb{R}^2$ be a fixed disc of area $n$, and set $A_\lambda(B_n) = A_\lambda \cap B_n$. Write $B_\lambda(n)$ for the event that $A_\lambda(B_n)$ covers $B_n$ (except for the points of $P'$), and set $p_\lambda(n) = P(B_\lambda(n))$. Extending results in [7], we show that if $\lambda^3 n \log n \to \infty$, then $p_\lambda(n) \to 0$, while if $\lambda^3 n \log n (\log \log n)^2 \to 0$, then $p_\lambda(n) \to 1$.

1 Introduction

Place discs of radius $r$ in $\mathbb{R}^2$ so that their centers form a Poisson process of intensity 1, and let $B(n) \subset \mathbb{R}^2$ be a disc of area $n \gg r^2$. What is the probability that $B(n)$ is covered by the small discs? This question, inspired by biology [6], has a long history, and many detailed results are known about it [3, 5]. For instance, writing

$$\pi r^2 = \log n + \log \log n + t,$$

Svante Janson proved in 1986 [5] that coverage occurs with probability asymptotically $e^{-t}$, as $n \to \infty$. One approach to this result [1, 2] uses the fact that the obstructions to coverage are small uncovered regions, which essentially form their own Poisson process, of intensity $e^{-t}$. Although these uncovered regions may be of different shapes, they are all roughly the same size (with probability tending to one). Here we study a related natural coverage process, where there are many different potential obstructions of many different sizes.

To define the process, let $P$ and $P'$ be independent Poisson processes, of intensities 1 and $\lambda$ respectively, in $\mathbb{R}^2$. We will call the points of $P$ black points and the points of $P'$ red points. Place an open disc $D(p, r_p)$ of radius $r_p$ around each black point $p \in P$, where $r_p$ is maximal so that $D(p, r_p) \cap P' = \emptyset$. In other words, $r_p$ is the distance from the black point $p$ to the nearest red point $p' \in P'$. 

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to $p$. $p'$ is almost surely unique, and we will refer to it as the stopping point of the disc centered at $p$, or of $p$ itself. We thus obtain a random set $A_\lambda \subset \mathbb{R}^2$ which is the union of discs centered at the points of $\mathcal{P}$. Now let $B_n \subset \mathbb{R}^2$ be a fixed disc of area $n$, and set $A_\lambda(B_n) = A_\lambda \cap B_n$. Write $B_\lambda(n)$ for the event that $A_\lambda(B_n)$ covers $B_n$ (except for the points of $\mathcal{P}'$), and set $p_\lambda(n) = \mathbb{P}(B_\lambda(n))$.

Since adding red points makes coverage less likely, $p_\lambda(n)$ is a non-increasing function of $\lambda$, for fixed $n$. In addition, $p_\lambda(n)$ is non-increasing in $n$, with $\lambda$ fixed, because increasing $n$ corresponds to examining the random set $A_\lambda$ over a larger area.

This model, based on the secrecy graph [4], was inspired by the issue of security in wireless networks, and was studied in [7]. Our ultimate goal is to determine $p_\lambda(n)$ asymptotically. In this paper, we refine the results in [7] to provide a new condition under which $p_\lambda(n)$ tends to zero, and a corresponding new condition under which $p_\lambda(n)$ tends to one. These can be thought of as improved upper and lower bounds, respectively, on the value of $\lambda$ such that $p_\lambda(n) = 1/2$, and will be referred to as such in what follows. Due to the divergence of a certain integral, the proofs strongly suggest that there are obstructions on a range of scales, and we hope to investigate this further in the future.

## 2 Upper bound

**Theorem 1.** If $\lambda^3 n \log n \to \infty$, then $p_\lambda(n) \to 0$.

**Proof.** Our strategy will be to show that, under the hypothesis, the expected number of good configurations (defined below) tends to infinity. A routine application of the second moment method then shows that a good configuration occurs with high probability (probability tending to one). Finally, we show that a good configuration results in an uncovered region of $B_n$.

First, therefore, we define a good configuration. Such a configuration consists of an ordered triple $(p_1, p_2, p_3)$ of red points in $B_n$. $p_1$ and $p_2$ must lie at distance $t$, where $n^{-1/12} < t < 1$. $p_3$ must lie at distance between $50/t$ and $100/t$ of $p_1$, in such a way that the angle $p_3p_1p_2$ is between $\pi/4$ and $3\pi/4$. (The choice of these angles is somewhat arbitrary: all we need is that the angle $p_3p_1p_2$ is bounded away from 0 and $\pi$.) Write $\ell_{ij}$ for the perpendicular bisector of $p_i p_j$, and $S$ for the bi-infinite strip of width $||p_1 - p_2||$ centered on $\ell_{12}$. For ease of explanation, suppose that the segment $p_1p_2$ is horizontal, so that $S$ is vertical, and that $p_3$ lies above the line through $p_1$ and $p_2$. $\ell_{13}$ and $\ell_{23}$ intersect the boundary $\partial S$ of $S$ in four points; suppose that the highest of these lies at height $h \leq 110/t$ above $p_1p_2$. Write $R \subset S$ for the rectangle with base $p_1p_2$ and height $2h$ (containing all four intersections above), and $R' \subset S$ for its reflection in $p_1p_2$.

A good configuration must also have no black points in the rectangular region $R \cup R'$. Note that the area of $R \cup R'$ is at most 440. Consequently, writing $X$ for the number of good configurations, there exist absolute constants $C$ and $C'$ such that

$$\mathbb{E}(X) \sim C \int_{n^{-1/12}}^1 \lambda n \cdot \lambda t^{-2} \cdot \lambda t \, dt = C' \lambda^3 n \log n \to \infty.$$
Second, we show that we can apply the second moment method to prove that, with high probability, $X \geq 1$. For this to work, we require an upper bound on $\lambda$; it will suffice to assume $\lambda^4 n \to 0$. Since $p_\lambda(n)$ is decreasing in $\lambda$, if we can prove that $p_\lambda(n) \to 0$ under the more restrictive hypotheses, the full result will follow. Tessellate $B(n)$ with squares of side length $n^{1/6}$, and color a square black if both of its “coordinates” are even. (Thus one out of every four squares is black.) We will only consider the black squares, which we label $S_1, S_2, \ldots, S_N$. Let the apex of a good configuration be the point furthest from the opposite side ($p_3$, in the above notation), and write $X_i$ for the number of good configurations with apex in $S_i$. With high probability, each $X_i$ will be either zero or one. Moreover, since the maximum diameter of a good configuration is $O(n^{1/12})$ by construction, the $X_i$ are independent. Let $X' = \sum X_i$. Then $E(X') \to \infty$ as above, and since
\[
P(X_i \geq 1) = O(\log n / n^{2/3}) \to 0,
\]
it follows that
\[
\text{Var}(X') = \sum \text{Var}(X_i) \sim \sum E(X_i) = E(X'),
\]
and so
\[
P(\text{Var}(X') \leq \text{Var}(X') \leq \frac{\text{Var}(X')}{E(X')^2} \sim \frac{1}{E(X')} \to 0.
\]

Finally, we explain why the presence of a good configuration prohibits full coverage. As above, suppose that $p_1 p_2$ is horizontal, and that $p_3$, and hence $\ell_{13}$ and $\ell_{23}$, lie above $p_1 p_2$. The idea is that part of $\ell_{12}$ lying just above $p_1 p_2$ will be uncovered. Write $m_0$ for the midpoint of $p_1 p_2$, and $m_s$ for the point of $\ell_{12}$ at height $s$ above $p_1 p_2$. Any black points lying in $S$ and above $p_1 p_2$ are much closer to $p_3$ than to $p_1$ or $p_2$, and so their corresponding discs cannot cover $m_0$ or $m_s$, for $s \sim C/t$. Write $q_1$ for the intersection of $\ell_{13}$ with $\partial S$ lying above $p_1$, $q_3$ for the intersection of $\ell_{23}$ with $\partial S$ lying above $p_2$, and $q_3$ for the midpoint of the opposite side of $R'$ from $p_1 p_2$. The $q_i$ are the best locations to place black points for the purposes of covering points $m_s$, for small $s$. However, even their corresponding black discs fail to cover $m_{s'}$, for suitable $s'$. Specifically, for $i = 1, 2$, write
\[
D_i = D(q_i, r_{q_i}) = D(q_i, ||q_i - p_{2-i}||) = D(q_i, ||q_i - p_3||),
\]
and
\[
D_3 = D(q_3, r_{q_3}) = D(q_3, ||q_3 - p_1||) = D(q_3, ||q_3 - p_2||).
\]
If the distance of $q_i$ from $p_1 p_2$ is $c_i / t$, then the heights of $D_1$ and $D_2$ above $m_0$ are asymptotically $t^4 / 8c_1$, and $D_3$ only covers $m_s$ for $s < t^4 / 8c_3$ (asymptotically). However, by construction,
\[
c_3 \geq \frac{3}{4} \max\{c_1, c_2\},
\]
so the point $m_{s'}$, for $s' = t^4 / 7c_3$, will be uncovered by $D_1 \cup D_2 \cup D_3$, and hence by $A_\lambda$. \qed
3 Lower bound

We require a lemma from [7]. This relates to the one-dimensional version of the problem, where we are covering an interval of length \(n\) with small intervals centered at black points, which in turn are stopped by red points. In [7], the lemma was used to show that, in one dimension, if \(\lambda \to \infty\) and also \(\lambda n \to \infty\), then (with obvious notation)

\[
p_{\lambda}(n) \sim e^{-4n\lambda^2}.
\]  

(1)

The lemma itself concerns an interval \(L\) of length \(\ell\) between two consecutive red points.

**Lemma 2.** Let \(L\) be an interval of length \(\ell\) between two consecutive red points. Let \(\mathcal{P}\) be a Poisson process of intensity 1 in \(L\), and grow an interval \(I_p\) centered at each point \(p\) of \(\mathcal{P}\) until it hits one of the red points at the endpoints of \(L\). Then

\[
P \left( L \text{ is covered by } \bigcup_{p \in \mathcal{P}} I_p \right) = 1 - e^{-\ell/2}(1 + \ell/2).
\]

**Proof.** Let \(m\) be the midpoint of \(L\), let \(x\) be the distance of the closest black point to \(m\) lying on the left of \(m\), and let \(y\) be the distance of the closest black point to \(m\) lying on the right of \(m\). Then coverage of \(L\) is determined solely by \(x\) and \(y\). Indeed, coverage occurs if and only if \(x + y \leq \ell/2\). Now \(x + y\) has the gamma distribution with density function \(te^{-t}\), and consequently

\[
P \left( L \text{ is covered by } \bigcup_{p \in \mathcal{P}} I_p \right) = \int_0^{\ell/2} te^{-t} \, dt = 1 - e^{-\ell/2}(1 + \ell/2),
\]

as required. \(\square\)

The deduction of (1) from Lemma 2 is straightforward. Firstly, the unconditional probability that the interval between two consecutive red points is covered, obtained by integrating the above probability against the density function of \(\ell\), is \((1 + 2\lambda)^{-2} \sim 1 - 4\lambda\). Second, since there are asymptotically \(n\lambda \to \infty\) intervals between consecutive red points, and coverage fails independently in each one with probability asymptotically \(4\lambda \to 0\), the number of failures is approximately Poisson with mean \(4n\lambda^2\), and (1) follows.

We return to the original two-dimensional problem, for which we have the following bound.

**Theorem 3.** If \(\lambda^3 n \log n (\log \log n)^2 \to 0\), then \(p_{\lambda}(n) \to 1\).

**Proof.** This is a refinement of the proof of Theorem 7 from [7]. Suppose that \(n \to \infty\) and also that \(\lambda^3 n \log n (\log \log n)^2 \to 0\). First, we show that we need only worry about coverage of parts of \(B(n)\) which are close (within distance \(\sqrt{8\log n}\)) to a red point. To do this, we tessellate \(B(n)\) with squares of side
length $r = \sqrt{\log n}$. The probability that any small square of the tessellation contains no black point is $e^{-\log n} = n^{-1}$. Since there are $\sim n / \log n$ such squares, the expected number of them containing no black points is asymptotically $1 / \log n \to 0$. Consequently, with high probability, every small square contains a black point. Now fix a small square $S$. If no point of $S$ is within distance $\sqrt{2 \log n}$ of a red point, and if $S$ contains a black point, then all of $S$ will be covered by $A_\lambda$. Therefore, with high probability, any point of $B(n)$ at distance more than $\sqrt{8 \log n}$ from all red points will be covered by $A_\lambda$, and we may assume this from now on.

It remains to show that the regions of $B(n)$ within distance $\sqrt{8 \log n}$ from a red point are covered by $A_\lambda$. Color such regions yellow. In order to facilitate a division into cases, let us construct a graph $G = G(n, P')$ on the red points by joining two red points if they lie within distance $R = R(n) = \sqrt{128 \log n}$ of each other. (Such a graph is usually called a random geometric graph.) A routine calculation shows that, with high probability, the connected components of $G$ consist of $o(n^{2/3}(\log n)^{-1/3})$ isolated vertices, $o(n^{1/3}(\log n)^{1/3})$ edges, $o(\log n)$ triangles, and $o(\log n)$ paths of length 2 (i.e., paths with 2 edges). We deal with each of these in turn; it will be convenient to consider a path of length 2 as a triangle, even though one of its edges is “long”.

**Isolated vertices.** Consider the circles of radii $\sqrt{8 \log n}$ and $\sqrt{32 \log n}$ around each isolated red point, and divide the annulus between these circles into 6 equal “sectors”, each of area $4\pi \log n$. With high probability, there is a black point inside each sector, and this black point is closer to the associated $p_i$ than to any other red point. Thus the yellow regions outside $S$ are covered by $A_\lambda$. However, coverage of the yellow regions inside the critical strip $S$ is not guaranteed. Indeed, from the upper bound argument in the previous section, such coverage is threatened by the presence of red points at distance $\sim C/t$ from $e$. $G$ contains edges almost as short as $n^{-1/6}$, so such points may lie almost as far as $n^{1/6}$ from $e$, almost as much as the typical distance between red points.

In connection with $e$, consider the region

$$R_1^+ = [0, t] \times [\sqrt{2 \log n}, \sqrt{2 \log n} + \log n / t] \subset S,$$

and its reflection in the $x$-axis

$$R_1^- = [0, t] \times [-\sqrt{2 \log n}, -\sqrt{2 \log n} - \log n / t] \subset S,$$

together with the much smaller regions

$$R_2^+ = [0, t] \times [\sqrt{2 \log n}, \sqrt{2 \log n} + 5 \log n / t] \subset S,$$
and
\[ R^-_2 = [0, t] \times [-\sqrt{2 \log n} - \sqrt{2 \log n - 5 \log \log \log n/t}] \subset S. \]

\( R_1^+ \) and \( R_1^- \) each have area \( \log n \), while \( R_2^+ \) and \( R_2^- \) each have area \( 5 \log \log n \).

Call an edge bad if some point of \( R_1^+ \cup R_1^- \) is closer to a third red point than to \( p_1 \) or \( p_2 \), very bad if some point of \( R_2^+ \cup R_2^- \) is closer to a third red point than to \( p_1 \) or \( p_2 \), and good otherwise. Write \( B \) for the number of bad edges, and \( V \) for the number of very bad edges. For an edge to be bad, there must be a third red point \( p_3 \) within distance \( C \log n/t \) of \( e \), and, for an edge to be very bad, \( p_3 \) must lie within distance \( C \log n/t \) of \( e \), for some absolute constant \( C \). (Note that, since \( e \) is an isolated edge, such a \( p_3 \) must lie at distance at least \( \sqrt{128 \log n} \) from both \( p_1 \) and \( p_2 \).) Moreover, the expected number of edges of length less than \( n^{-1/6} \) is \( O(\lambda n \cdot \lambda n^{-1/3}) = o(1) \), so, with high probability, there are no such edges. Consequently,

\[ E(B) \leq C' \int_{n^{-1/6}}^{\sqrt{128 \log n}} \lambda n \cdot \lambda (\log n)^2 t^{-2} \cdot \lambda t \, dt \leq C'' \lambda^3 n (\log n)^3 = o(\log n)^2, \]

and

\[ E(V) \leq C' \int_{n^{-1/6}}^{\sqrt{128 \log n}} \lambda n \cdot \lambda (\log n)^2 t^{-2} \cdot \lambda t \, dt \leq C'' \lambda^3 n \log n (\log n)^2 \to 0, \]

for some absolute constants \( C' \) and \( C'' \). Thus, with high probability, there are no very bad edges, and in fact most edges are good.

The idea behind the construction of \( R_1^+ \), \( R_1^- \), \( R_2^+ \) and \( R_2^- \) is that, in the absence of very bad edges, we may use the black points in \( R_1^+ \) and \( R_1^- \) to cover the yellow regions close to good edges, and those in \( R_2^+ \) and \( R_2^- \) to cover the yellow regions close to bad edges. For the purposes of covering such yellow regions in the critical strip \( S \), only the \( x \)-coordinates of black points in \( R_1^+ \), \( R_1^- \), \( R_2^+ \) and \( R_2^- \) matter. (The reason behind the \( \sqrt{2 \log n} \) term in the definitions of these rectangles is so that coverage of the entire yellow region follows from coverage of \( e \) from both sides.) Considering the good edges first, we project the black points in \( R_1^+ \) and \( R_1^- \) to the edge \( e = [0, t] \), where they form two separate Poisson processes, each of intensity \( \log n/t \), on an interval of length \( t \). For coverage purposes, these are equivalent to two processes of intensity 1 on an interval of length \( \log n \), and so we see from the lemma that full coverage of the relevant yellow regions fails with probability at most

\[ 2e^{-\log \log n/2(1 + \log n/2)} < \frac{2 \log n}{\sqrt{n}}. \]

Since we only expect \( o(n^{1/3}(\log n)^{1/3}) \) edges in \( G \), the yellow regions close to good edges are covered with high probability. Turning to the bad edges, and performing a corresponding projection of the black points in \( R_2^+ \) and \( R_2^- \), we see that this time coverage fails with probability at most

\[ 2e^{-5 \log \log n/2(1 + 5 \log n/2)} < \frac{10 \log \log n}{(\log n)^5/2}, \]
and, since we only expect $o((\log n)^2)$ bad edges in $G$, the yellow regions close to bad edges are also covered with high probability. Consequently, with high probability, all yellow regions near all edges of $G$ inside $B(n)$ are fully covered.

**Triangles.** We expect $o((\log n)^2)$ of these in $G$. The expected number of them with one edge shorter than $100 \log \log n$ is $O(\lambda^3 n (\log n)(\log \log n)^2) = o(1)$, so we may assume that each edge in each such triangle $T$ is longer than $100 \log \log n$. For simplicity, we deal with the case where each angle of $T$ is less than $80^\circ$, the other cases being, if anything, easier. Let the vertices of $T$ be $p_1, p_2$ and $p_3$, and let the circumcenter of $T$ be $c$. Let $Q_i$ be the rectangle with base $p_{i+1}p_{i+2}$, whose opposite side $s$ from $p_{i+1}p_{i+2}$ contains $c$ (here, subscripts are taken modulo 3), and let $Q'_i$ be the “top half” of $Q_i$, whose base is halfway from $p_{i+1}p_{i+2}$ to $s$, and whose opposite side still contains $c$. Projecting the $\Omega((\log \log n)^2)$ black points in $Q'_i$ to $p_{i+1}p_{i+2}$ as before, we see that the interior of $T$ is covered by $A_\lambda$ with probability $1 - o((\log n)^{-2})$, so that the interiors of all such triangles are covered with high probability. The exteriors of the triangles are also covered with high probability (this can be seen using such “sectors” as were used in the case of isolated vertices).

Consequently, with high probability, $A_\lambda$ covers all yellow regions in $B(n)$. Thus, also with high probability, $B(n)$ itself is covered.

I suspect that the lower bound is closer to the truth.

**References**


