Recall that the number of ways of choosing $r$ objects from $n$ (without replacement, where the order does not matter) is denoted $\binom{n}{r}$ and pronounced “$n$ choose $r$”. These binomial coefficients have many properties, for instance

- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \leq r \leq n - 1$
- $(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$
- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- $p | \binom{n}{r}$ if $p$ is prime and $1 \leq r \leq p - 1$

As an example of some of these facts in action, we can prove that

- $\sum_{r=0}^{s} \binom{m}{r} \binom{n}{s-r} = \binom{m+n}{s}$

The Fibonacci numbers are defined by

- $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

You should prove, for instance, that the number of ways of tiling a $1 \times n$ rectangle with squares and “dominoes” is $F_{n+1}$. There is a connection between Fibonacci numbers and binomial coefficients, namely,

- $F_n = \sum_{r} \binom{n-r}{r-1}$

which can be useful and which you should prove by induction.
Examples

1. (Putnam 1990) If $X$ is a finite set, let $|X|$ denote the number of elements in $X$. Call an ordered pair $(S, T)$ of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer.

2. (Putnam 1991) Suppose $p$ is an odd prime. Prove that

\[
\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.
\]

3. (Putnam 1992) For nonnegative integers $n$ and $k$, define $Q(n, k)$ to be the coefficient of $x^k$ in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

\[
Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j}.
\]

4. (Putnam 1996) Define a *selfish* set to be a set which has its own cardinality (number of elements) as a subset. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

5. (Putnam 2000) Prove that the expression

\[
\frac{\gcd(m, n)}{n} \binom{n}{m}
\]

is an integer for all pairs of integers $n \geq m \geq 1$.

Homework

1. (Putnam 1985) Determine, with proof, the number of ordered triples $(A_1, A_2, A_3)$ of sets which have the property that

\[
A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},
\]

and

\[
A_1 \cap A_2 \cap A_3 = \emptyset,
\]

where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where $a, b, c,$ and $d$ are nonnegative integers. [Hint. Draw a Venn diagram.]
2. (Putnam 1987) Let $r, s$ and $t$ be integers with $0 \leq r, 0 \leq s$, and $r + s \leq t$. Prove that

\[
\binom{s}{0} \binom{t}{r} + \binom{s}{1} \binom{t}{r+1} + \ldots + \binom{s}{s} \binom{t}{r+s} = \frac{t+1}{(t+1-s)(t-r)}.
\]

[Hint. Use induction on $s$.]