Constructing infinite dimensional polytopes using Gale transforms

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Abstract

A method already exists for constructing the Gale transform of an infinite-dimensional α-polytope with countably many extreme points. This transform can be used to determine all closed faces of the polytope. In this paper we study the reverse process, showing that the Gale transform can be used to construct an α-polytope.

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1 Introduction

Gale diagram techniques, first described in detail by Grünbaum [3], have proved a useful tool in the study of convex polytopes in finite dimensions. In particular, the combinatorial properties of the transform determine those of the polytope. These ideas were extended in Kleinschmidt and Wood [5], where it was shown that Gale transforms can be constructed for certain infinite dimensional polytopes.

The usefulness of the Gale transform in finite dimensions arises from the fact that, given a transform, it is possible to construct a polytope with this transform: the polytope-transform route is two-way. In this paper we uncover the return route from transform to infinite-dimensional polytope (Theorem 1), and thereby establish a method for constructing non-trivial infinite-dimensional polytopes.

An interesting contrast with the finite-dimensional case emerges. Since the Gale transform contains only algebraic information about the polytope, we find that a single transform can give rise to topologically distinct polytopes (Example 1).

The format of the paper is as follows. In §2 we set up the structure which takes us from transform to polytope. This is an extension of the coordinate free structure introduced in [5], and is centred on a pair of linear maps, and their dual sequence. It depends critically on the construction of a "Gale basis". A method for constructing such a basis is given in §3. We conclude the paper in §4 with a number of examples which illustrate the construction process. All notation and terminology used is as in [5], which in turn is based on the survey paper of McMullen, [7].

2 From transform to polytope

In order to construct the Gale transform in [5] we restricted ourselves to α-polytopes, in Banach spaces, with countably many extreme points. An α-polytope K is a compact convex set whose linear space of affine dependences, $N(K)$, is finite dimensional (Alfsen [1], Phelps [9]). The construction began with a countable, compact subset $X = \{x_n : n \in \mathbf{N}\}$ in a real Banach space E. The set X constituted the extreme points $\partial_e K$ of the α-polytope $K = \bar{\text{co}}\ X$. Thus

$$N(K) = \left\{ (\alpha_n) \in l_1 : \sum_{n=1}^{\infty} \alpha_n x_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 0 \right\}$$

has finite dimension, d say. Our analysis hinged around a sequence of Banach spaces and continuous linear maps between them, namely

$$\mathbf{R}^d \xrightarrow{g} l_1 \xrightarrow{r} E ,$$

in which the resultant map r is defined by $r(\delta_n) = x_n$ for all $n \in \mathbf{N}$, where δ_n is the sequence whose nth term is one, and all other terms are zero. The sequence is exact at l_1, so that $\mathcal{R}(g) = \mathcal{N}(g) = N(K)$. Here $\mathcal{R}(g)$ denotes the range of g, and $\mathcal{N}(r)$ denotes the null space of r. The dual sequence is

$$\mathbf{R}^d \xleftarrow{g^*} l_\infty \xleftarrow{r^*} E^* ,$$

where the weak*-closure of $\mathcal{R}(r^*)$ equals $\mathcal{N}(g^*)$. The Gale transform $\bar{X} \subseteq \mathbf{R}^d$ of X was then defined as $\{g^*(\delta_n) : n \in \mathbf{N}\ \}$.

In this paper we shall begin with the transform $\bar{X}$, construct the second sequence of maps, and recover the first. Specifically, we set up a sequence

$$\mathbf{R}^d \xleftarrow{f} l_\infty \xleftarrow{q} l_\infty ,$$

where $\bar{X} \subseteq \mathbf{R}^d$ and weak*-cl $\mathcal{R}(q) = \mathcal{N}(f)$. We then dualise to recover the first sequence, construct X as $\{q^*(\delta_n) : n \in \mathbf{N}\}$, and show it to constitute the extreme points of an α-polytope with Gale transform $\bar{X}$.

There are three critical contributions made in this paper which enable us to move from transform to polytope:

(i) Characterisation of the Gale transform; this enables us to build f.

(ii) Construction of a "Gale basis" for the null space of f; this enables us to build q. It is analogous to the construction of a basis for the affine dependences of X, upon which the formation of $\bar{X}$ relies.

(iii) Use of a sequence space duality first developed in the 1930's; this enables us to recover the required dual sequence of maps. This duality is well-established in the literature, being discussed for example in Kamthan and Gupta [4], Ruckle [10] and Wilansky [12], and coincidentally termed α-duality.

We proceed to the details of this construction of an α-polytope, starting with a Gale transform.

2.1 Gale transforms

Gale transforms of α-polytopes are characterised in the following way.

Definition 1 *A Gale transform, for an α-polytope with countably many extreme points and affine dependences of dimension d, is a countable set $\bar{X} = \{\bar{x}_n : n \in \mathbf{N}\}$ spanning $\mathbf{R}^d$ such that* *(i) every open half space* H^+ *of* $\mathbf{R}^d$ *with* $0 \in H$ *contains at least two points of* $\bar{X}$*,*

(ii) $\sum\limits_{n=1}^{\infty} \bar{x}_n = 0$*, and*

(iii) $\sum\limits_{n=1}^{\infty} \|\bar{x}_n\|_1 < \infty$.

Remarks

(a) No two of these conditions imply the third.

(b) We can view the Gale transform as a $d \times \aleph_0$ matrix, F, with $\bar{x}_n = (\bar{x}_{1n}, \ldots, \bar{x}_{dn})'$ the nth column. That is

$$F = \begin{bmatrix} \bar{x}_{11} & \ldots & \bar{x}_{1n} & \ldots \\ \vdots & & \vdots & \\ \bar{x}_{d1} & \ldots & \bar{x}_{dn} & \ldots \end{bmatrix}$$

Then (i) ensures that the rows are linearly independent, (ii) that the sum of the columns is $0 \in \mathbf{R}^d$ and (iii) that each row is an element of l_1.

(c) Condition (iii) was not brought out explicitly in [5], but does follow for the Gale transforms developed there. In that paper, the d elements of l_1, $\nu_1, \ldots, \nu_d$, which formed a basis for the affine dependences of the compact convex set K, form the rows of the F matrix. Since $\sum\limits_{i=1}^{d} \|\nu_i\|_1 < \infty$, (iii) follows.

2.2 The first map, f

Given a Gale transform, $\bar{X} \subseteq \mathbf{R}^d$, the matrix F determines a linear map f from l_∞ to $\mathbf{R}^d$. Specifically, for each $(\beta_n) \in l_\infty$ we define $f[(\beta_n)] = \sum\limits_{n=1}^{\infty} \beta_n \bar{x}_n$. Note that the d component functions of f are just the continuous linear functionals on l_∞ determined by the rows ν_i of F, for $i = 1, \ldots, d$. We summarise relevant properties of f in the following proposition.

Proposition 1 *The map $f : l_\infty \to \mathbf{R}^d$ determined by $\bar{X}$ has the following properties:*

(i) f *is linear and continuous from* l_∞ *with the weak* topology to* $\mathbf{R}^d$ *with the usual topology,*

(ii) $f(\delta_n) = \bar{x}_n$*, for each* n*,*

(iii) *(a)* $\mathcal{N}(f)$*, the null space of* f*, is* $\{(\beta_n) \in l_\infty : \sum\limits_{n=1}^{\infty} \beta_n \bar{x}_n = 0\} =$ *span* $\{\nu_1, \ldots, \nu_d\}^\perp$*, where* $M^\perp$ *is the annihilator in* l_∞ *of the subspace* M *in* l_1*.*

(b) $\mathcal{N}(f)$ *is weak*-closed and* $\underline{1} \in \mathcal{N}(f)$*, where* $\underline{1} = (1, 1, 1, \ldots)$*.*

Proof: Evidently f is linear, and since the weak* topology on l_∞ is the weakest for which each component functional is continuous, property (i) follows. Property (ii) is immediate from the definition of f. Property (iii) (a) follows from the definition of the annihilator. That $\mathcal{N}(f)$ is weak*-closed follows from (i), and that $\underline{1} \in \mathcal{N}(f)$ follows from the second condition= on $\bar{X}$.

For $(\beta_n) \in l_\infty$, we term $\sum_{n=1}^{\infty} \beta_n \bar{x}_n$ an l_∞*-linear combination of* $(\bar{x}_n)$,

2.3 Gale bases

Critical to our construction of the (f, q) sequence is the setting up of a countable subset within $\mathcal{N}(f)$, having certain prescribed properties. We term this subset a Gale basis.

Definition 2 *Let N be a subspace of l_∞ with finite codimension. A Gale basis for N is a matrix Q of the form*

$$Q = \begin{bmatrix} x_{11} & \dots & x_{1k} & \dots \\ \vdots & & \vdots & \\ x_{n1} & \dots & x_{nk} & \dots \\ \vdots & & \vdots & \end{bmatrix}$$

such that

(a) the rows, $x_1, x_2, \dots$ form a compact subset of distinct elements of l_1, and

(b) the weak-closure of all l_∞-combinations of the columns equals N.*

The columns of Q form the elements of the Gale basis. A method for constructing a Gale basis is given in §3.

2.4 The second map, q

Recall that $\mathcal{N}(f)$, the null space of f, has finite codimension, d (Proposition 1(iii)(a)). Then Q, a Gale basis for $\mathcal{N}(f)$, determines a linear map q from l_∞ to l_∞ as follows. For each $\underline{\beta} = (\beta_n) \in l_\infty$, define

$$q(\underline{\beta}) = (\gamma_n) \ , \quad \text{where} \ \ \gamma_n = \sum_{k=1}^{\infty} \beta_k x_{nk} \ , \quad \text{for all} \ \ n \in \mathbf{N} \ .$$

Condition (a) in the Gale basis definition ensures that the rows of Q are bounded in l_1. This ensures that q maps l_∞ into l_∞ (Petersen [8, Theorem 1.3.2]). The next proposition summarises the relevant properties of q.

Proposition 2 *The map $q : l_\infty \to l_\infty$ determined by Q has the following properties:*

(i) q is linear, and continuous from l_∞ with the strong topology to l_∞ with the strong topology.

(ii) $\text{weak}^*\text{-cl}\,\mathcal{R}(q) = \mathcal{N}(f)$ $(= N)$.

Proof: It is immediate that q is linear, while for $\underline{\beta} \in l_\infty$ with $\|\underline{\beta}\|_\infty = 1$, we have

$$\begin{aligned} \|q(\underline{\beta})\|_\infty &= \sup_n |\gamma_n| = \sup_n \Big| \sum_{k=1}^{\infty} \beta_k x_{nk} | \\ &\leq \sup_n \sum_{k=1}^{\infty} |\beta_k x_{nk}| \\ &\leq \sup_n \sum_{k=1}^{\infty} |x_{nk}| < M < \infty , \end{aligned}$$

since the rows of Q form a compact set. Thus q is bounded, hence continuous with respect to the norm topologies, so (i) holds. Definition 2(b) ensures that (ii) holds, since $\mathcal{R}(q)$ coincides with all l_∞-combinations of the columns of Q.

To summarise, using $\bar{X}$ we have constructed continuous linear maps f and q,

$$\mathbf{R}^d \xleftarrow{f} l_\infty \xleftarrow{q} l_\infty ,$$

such that weak*-cl $\mathcal{R}(q) = \mathcal{N}(f)$.

2.5 Sequence spaces and α-duality

The topological dual of l_∞ is a space very much larger than l_1. This appears to block the route from Gale transform to α-polytope; forming the topological dual of the (f, q) sequence does not return us to the (g, r) sequence of [5]. An alternative notion of duality is available, however, providing us with the mechanism and theory we need. The authors are most grateful to Professor Lee Peng Yee for alerting us to this duality. It was introduced by Köthe & Toeplitz in [6], to ensure that the dual of a sequence space is again a sequence space, precisely our need here. Our standard reference on α-duality is [4], with support from [10] and [12]. We pause to recall the definition of the α-dual of a sequence space, and to summarise the properties of the duality which we require.

Any subspace E of the space ω of all real sequences is termed a (real) sequence space. For such a space the α-dual E^α, sometimes called the Köthe-Toeplitz dual, is defined as

$$E^\alpha = \{(\alpha_n) \in \omega \ : \ \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty \ \text{ for all } \ (\beta_n) \in E\} \ .$$

With this definition $(l_\infty)^\alpha = l_1$ and $(l_1)^\alpha = l_\infty$. For these results and a full discussion of these ideas we refer the reader to [4, Chapter 2, §1 and §2].

Topologies on the α-dual pair (l_1, l_∞) can be determined as polar topologies in the usual way. The weak topology on l_1 determined by l_∞, $\sigma(l_1, l_\infty)$, is the weakest topology for which all functionals on l_1 given by elements of l_∞ remain continuous. Similarly the weak topology $\sigma(l_\infty, l_1)$ is the weak* topology on l_∞ in the customary language of topological duality. The strong (or norm) topologies on l_1 and $l_\infty, \beta(l_1, l_\infty)$ and $\beta(l_\infty, l_1)$ respectively, are compatible with the dual pair in that, for example, the continuous linear functionals on l_∞ with the norm topology include those determined by elements of l_1. We shall need just these topologies in our analysis.

A matrix transformation $A = (a_{nk})$ from a sequence space E to itself is a linear transformation of the sequence space determined by the infinite matrix, [4, p.205]. Such transformations are precisely those which are $\sigma(E, E^\alpha)$ to $\sigma(E, E^\alpha)$ continuous, [4, Proposition 3.2]. If we define $A^* = (a_{kn})$, the transpose of A, it follows that if A is a matrix transformation from E to itself, then A^* is a matrix transformation from E^α to E^α, [4, Proposition 3.3(i)]. With this background we are now able to dualise our (f, q) sequence appropriately.

2.6 The α-dual sequence

The proposition of this section presents the α-dual sequence of our (f, q) sequence.

Proposition 3 *Given a Gale transform, $\bar{X}$, construct the sequence*

$$\mathbf{R}^d \xleftarrow{f} l_\infty \xleftarrow{q} l_\infty \tag{1}$$

as described in Propositions 1 and 2. Consider the α-dual sequence

$$\mathbf{R}^d \xrightarrow{f^*(=g)} l_1 \xrightarrow{q^*(=r)} l_1 \ . \tag{2}$$

Then

(i) f^* and q^* are linear and continuous with respect to the strong topologies.

(ii) $\mathcal{R}(f^*) = \mathcal{N}(q^*) =$ span $\{\nu_1, \ldots, \nu_d\}$.

Proof: For (i), f^* is determined by F^*, and q^* by Q^*, so they are linear and weak-to-weak continuous maps. Since the weak and strong topologies coincide in both $\mathbf{R}^d$ and l_1 (Dunford and Schwartz [2, p.296]) the result follows.

We now address (ii). If we take the topological dual of sequence (2) we return to sequence (1), so we are able to draw on standard results from the theory of topological duality. Since f^* and q^* recapture the maps g and r respectively of [5], we simplify matters by denoting f^* as g and q^* as r (f and q were chosen as the predecessors of g and r in the alphabet).

Now

$$\begin{aligned}
\mathcal{R}(g)^{\perp} &= \mathcal{N}(f)\ , && \text{[10, Theorem 4.12]},\\
&= \text{weak}^* - \text{cl}\ \mathcal{R}(q)\ , && \text{by construction},\\
&= ({}^{\perp}\mathcal{R}(q))^{\perp}\ , && \text{[10, Theorem 4.7(b)]},\\
&= \mathcal{N}(r)^{\perp}\ , && \text{[10, Theorem 4.12]}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{R}(g) &= \text{cl}\ \mathcal{R}(g)\ , && \text{since } \mathcal{R}(g) \text{ is finite-dimensional},\\
&= {}^{\perp}(\mathcal{R}(g)^{\perp})\ , && \text{[10, Theorem 4.7(a)]},\\
&= {}^{\perp}(\mathcal{N}(r)^{\perp})\ , && \text{from the above calculation},\\
&= \text{cl}\ \mathcal{N}(r)\ , && \text{[10, Theorem 4.7(a)]},\\
&= \mathcal{N}(r)\ , && \text{since } \mathcal{N}(r) \text{ must be finite dimensional}.
\end{aligned}$$

Finally,

$$\mathcal{R}(g) = {}^{\perp}(\mathcal{R}(g)^{\perp}) = {}^{\perp}\mathcal{N}(f) = \text{span}\ \{\nu_1, \ldots, \nu_d\}$$

using Proposition 1(iii)(a).

2.7 Creating the α-polytope

We begin by using $\bar{X}$ to construct a countable set of points in l_1. In the theorem of this section we show these points to be the extreme points of an α-polytope with Gale transform $\bar{X}$.

Definition 3 *Given a Gale transform $\bar{X}$ and sequences (1) and (2) as in Proposition 3, define X to be the compact and countable set $\{r(\delta_n) : n \in \mathbf{N}\}$ in l_1.*

Note that since r is determined by Q^*, the elements of X are the rows of Q. Thus $X = \{x_n : n \in \mathbf{N}\}$.

We now present the main result.

Theorem 1 *Let $\bar{X}$ be a Gale transform. Then X, given in Definition 3, is such that*

(i) $K = \bar{co}\ X$ is an α-polytope,

(ii) $\bar{X}$ is a Gale transform for X, and

(iii) $X = \partial_e K$, the extreme points of K.

Proof: For (i), $K = \bar{\text{co}}\ X$ is compact and convex, and since the affine dependences of K, $\mathcal{N}(r)$, are spanned by the d linearly independent vectors $\nu_1, \ldots, \nu_d$, the result follows.

For (ii), given X, set up sequences (1) and (2) as in [5], with $\{\nu_1, \ldots, \nu_d\}$ a basis for $\mathcal{N}(r)$. Then $\{g^*(\delta_n) : n \in \mathbf{N}\ \}$ is a Gale transform for X, [5, Definition 1], and equals $\bar{X}$.

Finally, since the elements of X are the rows of Q, they are distinct, and so $\partial_e K = X$ provided each element x of X is an extreme point of K. Singletons $\{x\}$ in X are certainly closed in X, and since every open half-space H^+ of $\mathbf{R}^d$ with $0 \in H$ contains at least two points of $\bar{X}$, $0 \in$ *rel int* $\bar{co}\ \bar{X} \backslash \{\bar{x}\}$, and so $\bar{X} \backslash \{\bar{x}\}$ is a coface, [5, Definition 2 and Proposition 1]. Then we can invoke [5, Theorem 1(iii)] to conclude that x is extreme in K. Here we use the fact that the development of [5] remains valid for any countable and compact subset X of l_1 (see [5, Remark (ii)]).

We conclude by remarking that r is the resultant map determined by X.

3 Construction of the Gale basis

A problem which we passed over in the previous section was that of the construction of a Gale basis. We now address this critical problem. Examples 1(a) and 1(b) of §4 illustrate the construction process explained here so we suggest that the reader work through these examples in conjunction with this section.

Let $\bar{X}$ be a Gale transform, F be the matrix with rows $\nu_1, \ldots, \nu_d$ as in §2.1 and M be the linear span of $\{\nu_1, \ldots, \nu_d\}$. Given any natural number t we shall construct the infinite matrix Q (the Gale basis of Definition 2), with rows comprising t convergent sequences, by juxtaposing two matrices P and S. That is, $Q = [P \mid S]$. The matrix P has t columns and is such that the rows form a compact set of t sequences converging to $\{e_1,\ e_1 + e_2,\ e_1 + e_3, \ldots, e_1 + e_t\}$ where $\{e_1, \ldots, e_t\}$ is the standard basis for $\mathbf{R}^t$. The matrix S is then constructed to ensure that the rows of Q are distinct members of l_1 and that they converge in the same manner as the rows of P. At the same time, all columns of Q are set so that they belong to $M^{\perp}$.

How is P constructed? The first column is a column of 1's. This is used in Proposition 5 to show that the pre-perp of the linear span of the columns of Q equals M, a property required in the proof of Theorem 2. The remaining $t-1$ columns are repeated blocks, each block being a stack of matrices $\mathbf{C}$, $\mathbf{0}$ and $\mathbf{I}$. The modulus of each component of $\mathbf{C}$ tends to zero, $\mathbf{0}$ is a matrix of 0's, and $\mathbf{I}$ is the $(t-1)$-square identity matrix. The **C**-**O**-**I** structure ensures that the columns of P are in $M^{\perp}$, and allows us to extend the rows of P to form t convergent sequences, together with their limit points, in Q. The $t-1$ sequences formed by the rows in Q associated with the $\mathbf{I}$ matrices will converge to the $t-1$ points $\delta_1+\delta_2$ to $\delta_1+\delta_t$, while all other rows will converge to δ_1. The construction does not hold when the dimension of *span* $\bar{X}$ is zero, but a similar, simpler approach suffices. Details of the general construction follow the next lemma.

Lemma 1 *(i) Given a Gale transform* $\bar{X}$ *there is a set of* d *indices,* J*, such that* $\{\bar{x}_n : n \in J\}$ *is a basis for* $\mathbf{R}^d$.

(ii) For each $j \in J$ *there is a* $k \neq j$ *such that* $\{\bar{x}_n : n \in J\backslash\{j\}\} \cup \{\bar{x}_k\}$ *is a basis for* $\mathbf{R}^d$.

Proof: Both statements follow straightforwardly from Definition 1(i).

3.1 Construction of P

For each positive integer t, we construct an $\aleph_0 \times t$ matrix P such that

(i) the rows of P form t convergent sequences and include their limit points $\{e_1,\ e_1+e_2,\ e_1+e_3,\ldots,e_1+e_t\} \subseteq \mathbf{R}^t$, and

(ii) the columns of P belong to $M^{\perp}$.

Let p_n denote the nth row of P and let J be as in Lemma 1(i). To each $j_i \in J$ there corresponds a k_i as in Lemma 1(ii), with the k_i not necessarily distinct. Without loss of generality, assume that $J=\{1,\ldots,d\}$ and $K=\{d+1,\ldots,l\}$; if not, renumber $\{\bar{x}_n\}$ accordingly. For $t=1$ put $p_n=e_1$ for all n. Then $\{p_n\}$ satisfies property (i) trivially, and property (ii) since $\sum \bar{x}_n = 0$. For $t>1$ let A_1 be the nonsingular $d \times d$ matrix with columns $\bar{x}_1,\ldots,\bar{x}_d$ and let B_1 be the $d \times (t-1)$ matrix with columns $\bar{x}_{l+2},\ldots,\bar{x}_{l+t}$. Gale point $\bar{x}_{l+1}$ is avoided at this stage. The reason for this becomes evident during the proof of Proposition 4. Put

$$C_1 = -A_1^{-1}B_1 \quad , \tag{3}$$

so, for example, the first column of C_1 contains the coefficients of the linear combination of $\bar{X}$ yielding $\bar{x}_{l+2}$. Define the first $l+t$ rows of P as

$$\left[\begin{array}{c|c} \mathbf{1} & C_1 \\ & \mathit{0} \\ & I \end{array}\right] \tag{4}$$

where $\mathbf{1}$ is a column of 1's, $\mathit{0}$ is an $(l-d+1) \times (t-1)$ matrix of 0's and I is the $(t-1)$-square identity matrix. If $(\beta_n)_{n=1}^{l+t}$ is the ith column of P defined thus far, with $2 \le i \le t$, and $\{e_i^*\}$ is the standard basis for $\mathbf{R}^{t-1}$ then

$$\begin{aligned} \sum_{n=1}^{l+t} \beta_n \bar{x}_n &= \sum_{n=1}^{d} (C_1 e_{i-1}^*)_n \bar{x}_n + \bar{x}_{l+i} \ , \quad \text{using (4)}, \\ &= A_1 C_1 e_{i-1}^* + \bar{x}_{l+i} \\ &= -B_1 e_{i-1}^* + \bar{x}_{l+i} \ , \qquad\qquad \text{using (3)}, \\ &= 0 \end{aligned} \tag{5}$$

from the definition of B_1. Let $K_1 = l + t$; the rows of P are defined as far as row K_1. We now construct the remaining rows of P in blocks similar to those in (4) and with a property analogous to (5).

Take (ψ_n) to be a strictly positive null sequence. Suppose that P is defined as far as row K_{k-1} for some $k > 1$. Let M_k be the linear span of $\{\bar{x}_n : n > K_{k-1}\}$.

If the dimension of M_k is not zero there is an $L_k > K_{k-1}$ such that $\{\bar{x}_n : K_{k-1} < n \le L_k\}$ spans M_k. Define A_k to be the $d \times (L_k - K_{k-1})$ matrix with columns $\bar{x}_n$, $K_{k-1} < n \le L_k$. Then $A_k : \mathbf{R}^{L_k - K_{k-1}} \to M_k \subseteq \mathbf{R}^d$ is onto M_k and so has a right inverse $E_k : M_k \to \mathbf{R}^{L_k - K_{k-1}}$. The solution of $A_k z = -\bar{x}_n$ is then given by $z = z_n = -E_k \bar{x}_n$.

Choose $R_k \geq L_k$ such that $\|z_n\|_\infty \leq \psi_k$ for all $n > R_k$, possible since $\bar{x}_n \to 0$. Define B_k to be the $d \times (t-1)$ matrix with columns $\bar{x}_n$, $R_k < n \leq R_k + t - 1$, and put $C_k = -E_k B_k$. Define rows $K_{k-1}+1$ to $K_k = R_k + t - 1$ of P as

$$\left[\begin{array}{c|c} \mathbf{1} & C_k \\ & \mathit{0} \\ & I \end{array}\right]$$

where $\mathbf{1}$ is a column of 1's, $\mathit{0}$ is an $(R_k - L_k) \times (t-1)$ matrix of 0's and I is the $(t-1)$-square identity matrix. Note that the definition of R_k ensures that every component u of C_k is such that $|u| \leq \psi_k$. A calculation analogous to that producing (5) shows that for every column $(\beta_n)_{n=1}^{K_k}$ of P, except the first,

$$\sum_{n=1}^{K_k} \beta_n \bar{x}_n = 0 \; . \tag{6}$$

If the dimension of M_k is zero, put $K_k = R_k = K_{k-1} + t$ and define the t rows $K_{k-1}+1$ to K_k as

$$\left[\begin{array}{c|c} \mathbf{1} & \mathit{0} \\ & I \end{array}\right]$$

where $\mathit{0}$ is a zero row and I is the $(t-1)$ square identity matrix. Since the dimension of M_k is zero, $\bar{x}_n = 0$ for all $n > K_{k-1}$ so (6) certainly holds. We continue this process inductively to define the matrix P.

We now show that P has properties (i) and (ii) described earlier. The rows $\{p_n\}$ of P can be partitioned into t subsets, each comprising a convergent sequence and its limit point. For every $k > 1$ and $i = 1, \ldots, t-1$, $p_{R_k+i} = e_1 + e_{i+1}$ and so such rows certainly converge to $p_{l+i+1} = e_1 + e_{i+1}$. We now show that the remaining rows converge to $p_{l+1} = e_1$. Let $p = (u_1, \ldots, u_t)$ be one such row. Then p is either e_1 or a row of C_k for some k. In the latter case, $|u_i| \leq \psi_k$ for each $i = 2, \ldots, t$, and since $\psi_k \to 0$ as $k \to \infty$, such rows converge to $p_{l+1} = e_1$. Hence the rows of P form t convergent sequences and include their limit points.

It remains to show that the columns of P are in $M^{\perp}$; this is equivalent to showing that for each column (β_n) of P, $\sum_{n=1}^{\infty} \beta_n \bar{x}_n = 0$. If (β_n) is the first column of P, then $\beta_n = 1$ for all n and $\sum_{n=1}^{\infty} \beta_n \bar{x}_n = \sum_{n=1}^{\infty} \bar{x}_n == 0$. Suppose that (β_n) is the ith column of P, for $2 \leq i \leq t$. If the dimension of M_k is zero for some k, then the result follows immediately from repeated application of (6). Suppose now that the dimension of M_k is non-zero for all k. For $T > K_1$, there is a k such that $K_k < T \leq K_{k+1}$. Then,

$$\begin{aligned}
\left\| \sum_{n=1}^{T} \beta_n \bar{x}_n \right\| &= \left\| \sum_{n=K_k+1}^{T} \beta_n \bar{x}_n \right\| , \qquad \text{using (6)}, \\
&\leq \sum_{n=K_k+1}^{K_{k+1}} \|\beta_n \bar{x}_n\| \\
&= \sum_{n=K_k+1}^{L_k} (C_k e_{i-1}^{*})_n \, \|\bar{x}_n\| + \|\bar{x}_{R_k+i-1}\| \\
&\leq |\psi_k| \sum_{n=K_k+1}^{L_k} \|\bar{x}_n\| + \|\bar{x}_{R_k+i-1}\|
\end{aligned}$$

which converges to zero as $k \to \infty$. Here $\{e_i^{*}\}$ is the standard basis for $\mathbf{R}^{L_k - K_{k-1}}$. Since $k \to \infty$ as $T \to \infty$ we have

$$\sum_{n=1}^{\infty} \beta_n \bar{x}_n = 0 \quad , \tag{7}$$

completing the construction of P.

3.2 Construction of S

We construct an infinite matrix R and put $S = RD$, where D is a diagonal matrix which ensures that the rows of S tend to zero. This in turn will ensure that the rows of Q follow the convergence of those of P. Columns of R are formed by writing each $-\bar{x}_n$, for $n \notin \{l+1, \ldots, l+t\}$, as a linear combination of the linearly independent Gale points $\bar{x}_1, \ldots, \bar{x}_d$. Each column of R then belongs= to $M^{\perp}$. Rows $l+1$ to $l+t$ in R are defined to be zero, as these rows are the limit points of the rows of Q. We must also ensure that the rows of S are distinct. The structure imposed on S is shown in Proposition 4 to imply that if two of the first d rows are equal then all except d of the Gale points lie in a hyperplane in $\mathbf{R}^d$. When $d \leq 3$ this contradicts the definition of a Gale transform. For $d > 3$, however, it is necessary to write $\bar{x}_n$, for each $n = 1, \ldots, d$, as a linear combination of d independent Gale points which exclude $\bar{x}_n$ itself. This is made possible using the Gale points indexed by the set $K = \{d+1, \ldots, l\}$, and motivates the construction leading to (8) below.

Let $v_1 = -A^{-1}\bar{x}_1$ where A is the non-singular matrix with columns $\bar{x}_2, \ldots, \bar{x}_d, \bar{x}_{d+1} = \bar{x}_{k_1}$ (recall that k_1 is defined using Lemma 1(ii)). That is, the components of v_1 are the coefficients arising when we express $-\bar{x}_1$ as a linear combination of $\bar{x}_2, \ldots, \bar{x}_d, \bar{x}_{d+1}$. Define the first column of R to be

$$(\delta_1 + v_{1,1}\delta_2 + \cdots + v_{1,d-1}\delta_d + v_{1,d}\delta_{d+1})' \tag{8}$$

where $v_{i,j}$ is the jth component of v_i. Similarly define columns $2, \ldots, d$. For example, v_2 is constructed by writing $-\bar{x}_2$ as a linear combination of $\bar{x}_1, \bar{x}_3, \ldots, \bar{x}_d, \bar{x}_{k_2}$. For $i = d+1, \ldots, l$ let $v_i = -A_1^{-1}\bar{x}_i$ where A_1 is as in (3). Define the ith column of R to be

$$(v_{i,1}\delta_1 + \cdots + v_{i,d}\delta_d + \delta_i)' \tag{9}$$

For $i > l$ let $v_i = -A_1^{-1}\bar{x}_{t+i}$. Define the ith column of R= to be

$$(v_{i,1}\delta_1 + \cdots + v_{i,d}\delta_d + \delta_{t+i})' \; . \tag{10}$$

All rows of R are finitely non-zero except possibly the first d. If $K = \sum_{i=1}^{l} \|v_i\|$ then

$$\sum_{i=1}^{\infty} \|v_i\| \;=\; K + \sum_{i>l} \left\| -A_1^{-1} \bar{x}_{t+i} \right\| \;\leq\; K + \left\| -A_1^{-1} \right\| \sum_{i>l} \|\bar{x}_{t+i}\| \;<\; \infty \;,$$

and so the rows of R belong to l_1. Let D be a diagonal matrix with non-zero entries ϕ_i such that $\phi_i \to 0$ as $i \to \infty$. Put $S = RD$.

We claim that if (β_n) is the ith column of S, then

$$\sum_{n=1}^{\infty} \beta_n \bar{x}_n = 0 \;. \tag{11}$$

For example, if $i > l$ then (10) implies

$$\sum_{n=1}^{\infty} \beta_n \bar{x}_n \;=\; \phi_i \left(\sum_{n=1}^{d} v_{i,n} \bar{x}_n + \bar{x}_{t+i} \right) \;=\; \phi_i \left(A_1 v_i + \bar{x}_{t+i} \right) \;=\; 0 \;.$$

Similarly, if $i \leq d$ or $i \in \{d+1, \ldots, l\}$ then (8) and (9) give (11) respectively.

Finally, put $Q = \left[\; P \;\middle|\; S \;\right]$.

3.3 The infinite matrix Q forms a Gale basis for $N = M^{\perp}$

We reach this result via two propositions and a theorem.

Proposition 4 *The rows of Q are distinct.*

Proof: Items (4), (8), (9) and (10) ensure that rows from the $(d+1)$th onwards are distinct. For example, if $i \in \{l+1, \ldots, l+t\}$ and $j > l+t$ then row i is zero from the $(i-l)$th position onwards, while row j has a non-zero entry in the jth position, from (4) and (10). The first d rows are distinct from those beyond row l since by (8), if $1 \leq i \leq d$ and $j > l$ then in the $(t+i)$th position row i has a non-zero entry and row j has a zero entry, by (4) and (10). It remains to show that the first d rows are distinct, and that no row from the first d rows is equal to one from rows $d+1$ to l. Without loss of generality, suppose that row one equals row two. Then the first and second components of every column of Q are equal.

For $3 \leq n \leq d$, $\bar{x}_n$ is the nth column of A_1 and so $\left\langle -A_1^{-1}\bar{x}_n, e_1^*\right\rangle = 0 = \left\langle -A_1^{-1}\bar{x}_n, e_2^*\right\rangle$. Here $\{e_i^*\}$ is the standard basis in $\mathbf{R}^d$. ¿From (9), for $d < n \leq l$, and from (10), for $n > l+t$, we have $\phi_n \left\langle -A_1^{-1}\bar{x}_n, e_1^*\right\rangle = \phi_n \left\langle -A_1^{-1}\bar{x}_n, e_2^*\right\rangle$. ¿From the development of (4), for $l+2 \leq n \leq l+t$ we have $\left\langle -A_1^{-1}\bar{x}_n, e_1^*\right\rangle = \left\langle -A_1^{-1}\bar{x}_n, e_2^*\right\rangle$. Thus for all $n \notin \{1, 2, l+1\}$ we have $\left\langle \bar{x}_n, (-A_1^{-1})^*(e_1^* - e_2^*)\right\rangle = 0$, with $(-A_1^{-1})^*(e_1^* - e_2^*) \neq 0$ since A_1 has full rank. Then all but three Gale points lie in the same hyperplane passing through the origin, and so at least one of the open half-spaces defined by the hyperplane must contain one or no Gale point. This contradicts the definition of a Gale transform, and so rows one and two must be distinct. Similarly, if we assume that row one equals row $d+1$, we obtain $\left\langle \bar{x}_n, (-A_1^{-1})^* e_1\right\rangle = 0$ for all $n \notin \{1, d+1, l+1\}$, again a contradiction.

Proposition 5 *Let span* $Q \subseteq l_\infty$ *be the linear span of the columns of* Q. *Then*

$$M \; = \; {}^\perp(span\ Q) \; = \; \left\{(\alpha_n) \in l_1 \; : \; \sum \alpha_n \beta_n = 0 \; , \; \textit{for all columns} \;\; (\beta_n) \; \textit{of} \; Q\right\} \; .$$

Proof: Let (β_n) be a column of Q. Then for $i = 1, \ldots, d$, $\langle(\beta_n), \nu_i\rangle \; = \; \sum_{n=1}^{\infty} \beta_n \langle \bar{x}_n, e_i^*\rangle =$ 0 since $\sum_{n=1}^{\infty} \beta_n \bar{x}_n = 0$ using (7) and (11). Thus $M \subseteq {}^\perp(span\ Q)$.

We now show the reverse inclusion. Suppose that $(\gamma_n) \in {}^\perp(span\ Q)$, so that for the ith column (β_n) of Q, $\sum \beta_n \gamma_n$ is zero. Let $v = (A_1^{-1})^*(\gamma_1, \ldots, \gamma_d)' \in \mathbf{R}^d$, so that $\gamma_i = \langle \bar{x}_i, v\rangle$ for $i = 1, \ldots, d$.

If $t+d+1 \leq i \leq t+l$ then (β_n) is the $(i-t)$th column of S, and by (9),

$$\phi_{i-t}(v_{i-t,1}\gamma_1 + \cdots + v_{i-t,d}\gamma_d + \gamma_{i-t}) = 0, \quad \text{so that}$$

$$\gamma_{i-t} = -\sum_{j=1}^{d} \left\langle -A_1^{-1}\bar{x}_{i-t}, e_j^* \right\rangle \gamma_j = \langle \bar{x}_{i-t}, v \rangle \ . \tag{12}$$

If $i > t + l$ then by (10),

$$\phi_{i-t}(v_{i-t,1}\gamma_1 + \cdots + v_{i-t,d}\gamma_d + \gamma_i) = 0 \ , \quad \text{so that}$$

$$\begin{aligned} \gamma_i &= -\left(v_{i-t,1}\langle \bar{x}_1, v\rangle + \cdots + v_{i-t,d}\langle \bar{x}_d, v\rangle\right) \ , && \text{using (12),} \\ &= -\langle v_{i-t,1}\bar{x}_1 + \cdots + v_{i-t,d}\bar{x}_d, v\rangle \ = \ -\langle A_1 v_{i-t}, v\rangle \\ &= \langle \bar{x}_i, v\rangle \ , && \text{using (10).} \end{aligned} \tag{13}$$

If $2 \leq i \leq t$ then (β_n) is the ith column of P and

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \gamma_n \beta_n \\ &= \sum_{n=1}^{l} \langle \bar{x}_n, v\rangle \beta_n + \gamma_{l+i} + \sum_{n=l+t+1}^{\infty} \langle \bar{x}_n, v\rangle \beta_n \ , \quad \text{by (4) and (13),} \\ &= \gamma_{l+i} + \sum_{n=1}^{\infty} \langle \bar{x}_n, v\rangle \beta_n - \sum_{n=l+1}^{l+t} \langle \bar{x}_n, v\rangle \beta_n \\ &= \gamma_{l+i} + 0 - \langle \bar{x}_{l+i}, v\rangle \ . \end{aligned}$$

Thus

$$\gamma_{l+i} = \langle \bar{x}_{l+i}, v\rangle \ . \tag{14}$$

Finally, if (β_n) is the first column of Q then $\beta_n = 1$ for all n, and calculations similar to the above give $\gamma_{l+1} = \langle \bar{x}_{l+1}, v\rangle$. Combining this with (12), (13) and (14) we have $\gamma_n = \langle \bar{x}_n, v\rangle = v_1(\nu_1)_n + \cdots + v_d(\nu_d)_n$ for every n; that is $(\gamma_n) = v_1\nu_1 + \cdots + v_d\nu_d \in M$ so that ${}^{\perp}(\mathit{span}\ Q) \subseteq M$ and the proof of the proposition is complete.

Theorem 2 *The infinite matrix Q is a Gale basis for $N = M^{\perp}$.*

Proof: We are required to prove the following:

(i) the rows of Q are distinct and form a compact set in l_1, and

(ii) the weak*-closure of all l_∞-combinations of the columns of Q is N.

For (i), Proposition 4 ensures that the rows $\{q_n\}$ of Q are distinct. Recall that the rows $\{p_n\}$ of P are a union of convergent sequences, together with their limit points. Now consider each p_n to be an element of l_1 by adjoining a sequence of 0's. We show that if $\{p_{n_k}\}$ is a convergent subsequence of the rows of P then $\{q_{n_k}\}$ is convergent to the same limit in the rows of Q. Thus $\{q_n\}$ is a union of convergent sequences (with their limit points) and so is compact in l_1. Now

$$\begin{aligned}
\left\|q_{n_k} - \lim_k p_{n_k}\right\|_1 &\leq \|q_{n_k} - p_{n_k}\|_1 + \left\|p_{n_k} - \lim_k p_{n_k}\right\|_1 \\
&= \sum_{j=t+1}^{\infty} |(q_{n_k})_j| + \left\|p_{n_k} - \lim_k p_{n_k}\right\|_1 \\
&= |\phi_{n_k+d}| + \left\|p_{n_k} - \lim_k p_{n_k}\right\|_1, \qquad \text{for } n_k \text{ sufficiently large,}
\end{aligned}$$

which converges to zero as k tends to infinity. The limit points of the $\{p_{n_k}\}$ sequences are $p_{l+1}, \ldots, p_{l+t}$, and from the construction of S, $q_{l+1} = p_{l+1}, \ldots, q_{l+t} = p_{l+t}$, so these limit points belong to $\{q_n\}$.

For (ii), note that the rows of Q are compact in l_1, so are bounded in l_1. Thus Q satisfies the conditions in [8, p.5, Theorem 1.3.2] to be a matrix transformation of l_∞, and so l_∞-combinations of the columns converge. The set of all such l_∞-combinations is the range of Q, $\mathcal{R}(Q)$. We show $M = {}^\perp\mathcal{R}(Q)$. We have $M = {}^\perp(\mathit{span}\ Q)$ from Proposition 5, and since $\mathit{span}\ Q \subseteq \mathcal{R}(Q)$, $M = {}^\perp(\mathit{span}\ Q) \supseteq {}^\perp\mathcal{R}(Q)$. To show $M \subseteq {}^\perp\mathcal{R}(Q)$ take $(\gamma_n) \in l_\infty$ and note that for each ν_i, $i = 1, \ldots, d$,

$$\left\langle \nu_i, \sum_{n=1}^{\infty} \gamma_n \underline{\beta}_n \right\rangle = \sum_{n=1}^{\infty} \gamma_n \left\langle \nu_i, \underline{\beta}_n \right\rangle = 0$$

where $\underline{\beta}_n$ is the nth column of Q. Thus $\nu_i \in {}^{\perp}\mathcal{R}(Q)$ and so $M = {}^{\perp}\mathcal{R}(Q)$. Finally,

$$\begin{aligned} \text{weak}^*\text{-cl } \mathcal{R}(Q) &= \left({}^{\perp}\mathcal{R}(Q)\right)^{\perp}, && [11, \text{Theorem } 4.7(\text{b})], \\ &= M^{\perp}, && \text{since } {}^{\perp}\mathcal{R}(Q) = M, \\ &= N\ . \end{aligned}$$

4 Examples

Example 1. Let $\bar{X}$ be the (re-ordered) one-dimensional Gale transform of [5, Example 1]. That is,

$$\bar{X} = \{\bar{x}_1, \bar{x}_2, \ldots\} = \{1, -1, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{8}, \ldots\}.$$

We shall construct α-polytopes K_1 and K_3 each having Gale transform $\bar{X}$, K_1 having one limit point in its extreme boundary and K_3 having three limit points in its extreme boundary.

(a) Construction of K_1: Following the construction of §3, since $t = 1$ we put $P = (1, 1, \ldots)'$. The first two Gale points have the properties of Lemma 1(i) and (ii), and so $J = \{1\}$, $K = \{2\}$, $l = 2$ and $A_1 = \bar{x}_1 = 1$. We shall construct the matrix R. By= (9), the first column of R is $\delta_1 + \delta_2$ and from (9), column two is $\delta_1 + \delta_2$. For every $i > l = 2$, (10) gives $v_i = -\bar{x}_{t+i} = -\bar{x}_{1+i}$ so the ith column of R is $-\bar{x}_{1+i}\delta_1 + \delta_{1+i}$. Thus

$$R = \begin{bmatrix} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & \cdots \\ 1 & 1 & 0 & \cdots & & & & & \\ 0 & 0 & 0 & \cdots & & & & & \\ 0 & 0 & 1 & 0 & \cdots & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & & & \\ 0 & & \cdots & 0 & 1 & 0 & \cdots & & \\ \vdots & & & & & \ddots & & & \end{bmatrix}$$

Letting (ϕ_n) be given by $\phi_{2n-1} = \phi_{2n} = 1/2^{n-1}$ for $n \geq 1$ we obtain

$$Q = \left[\; P \mid S \;\right] = \begin{bmatrix} 1 & 1 & 1 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{64} & \frac{1}{64} & \cdots \\ 1 & 1 & 1 & 0 & \cdots & & & & & \\ 1 & 0 & 0 & 0 & \cdots & & & & & \\ 1 & 0 & 0 & \frac{1}{2} & 0 & \cdots & & & & \\ 1 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \cdots & & & \\ 1 & 0 & & \cdots & 0 & \frac{1}{4} & 0 & \cdots & & \\ 1 & 0 & & & \cdots & 0 & \frac{1}{4} & 0 & \cdots & \\ \vdots & & & & & & & \ddots & & \end{bmatrix}$$

The rows of Q form the set X_1, and $K_1 = \bar{co}\ X_1$ is the desired α-polytope.

(b) Construction of K_3: Here, the construction of P is fully illustrated. Choose $\psi_n = 1/n$ for all n. We have $t = 3$, and as in (a), $J = \{1\}$, $K = \{2\}$, $l = 2$, and $A_1 = \bar{x}_1 = 1$. Let $B_1 = [\bar{x}_4, \bar{x}_5] = [1/2, -1/2]$ and put $C_1 = -A_1^{-1}B_1 = [-1/2, 1/2]$. The first 5 rows of P are then

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The next "block" of P is constructed as follows: if $M_2 = \mathit{span}\ \{\bar{x}_n : n > 5\}$ then $\mathit{dim}\ M_2 = 1$ and $\{\bar{x}_6\}$ spans M_2 so $L_2 = 6$. Put $A_2 = \bar{x}_6$; the solution to $A_2 z = -\bar{x}_n$ is then $z = z_n = -4\bar{x}_n$. If $R_2 = 7$, then $|z_n| \leq \psi_2 = 1/2$ for all $n > R_2$. Put $B_2 = [\bar{x}_8, \bar{x}_9]$, and $C_2 = -4B_2 = [-1/2, 1/2]$. Rows 6 to 9 of P are then

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Continuing through the construction process we obtain $P' =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -\frac{1}{2} & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 1 & 0 & -\frac{1}{4} & 0 & 0 & 0 & 1 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{4} & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} & 0 & 0 & 0 & 0 & 1 & \cdots \end{bmatrix}$$

The construction of R is similar to that in (a). Taking (ϕ_n) as in (a) we obtain

$$Q = \left[\, P \mid S \,\right] = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & 1 & 1 & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{32} & \frac{1}{32} & -\frac{1}{128} & \frac{1}{128} & \cdots \\ 1 & 0 & 0 & 1 & 1 & 0 & \cdots & & & & & \\ 1 & 0 & 0 & 0 & \cdots & & & & & & & \\ 1 & 1 & 0 & 0 & \cdots & & & & & & & \\ 1 & 0 & 1 & 0 & \cdots & & & & & & & \\ 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \cdots & & & & \\ 1 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \cdots & & & \\ 1 & 1 & 0 & 0 & & \cdots & 0 & \frac{1}{4} & 0 & \cdots & & \\ 1 & 0 & 1 & 0 & & & \cdots & 0 & \frac{1}{4} & 0 & \cdots & \\ 1 & -\frac{1}{4} & \frac{1}{4} & 0 & & & & \cdots & 0 & \frac{1}{8} & 0 & \cdots \\ \vdots & & & & & & & & & & \ddots & \end{bmatrix}$$

Again, the rows of Q form the set X_3, and $K_3 = \bar{co}\ X_3$ is the desired α-polytope.

Example 2. Take $\bar{X}$ to be the re-ordered two-dimensional Gale transform of [5, Example 3]. That is,

$$\bar{x}_1 = (1,0), \qquad \bar{x}_2 = (0,1), \qquad \bar{x}_3 = (-1,-1), \qquad \bar{x}_4 = (0,0),$$

$$\bar{x}_{3k+2} = \left(\frac{1}{2^k}, 0\right), \qquad \bar{x}_{3k+3} = \left(0, \frac{1}{2^k}\right), \qquad \bar{x}_{3k+4} = \left(-\frac{1}{2^k}, -\frac{1}{2^k}\right)$$

for $k = 1, 2, \ldots$. We construct an α-polytope K_2 with $t = 2$ limit points in its extreme boundary X. Choose $\psi_n = 1/n$ for all n. The first two Gale points span $\mathbf{R}^2$, and the third has the property of Lemma 1(ii), so $J = \{1,2\}$, $K = \{3\}$ and $l = 3$. Now A_1 is the identity matrix and $B_1 = [\bar{x}_{l+2}] = [1/2, 0]'$ so following (3) we have $C_1 = -B_1 = [-1/2, 0]'$. The first 5 rows of P are then

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -\frac{1}{2} & 0 & 0 & 0 & 1 \end{bmatrix}'$$

For the next block of rows, $M_2 = span\ \{\bar{x}_n : n > 5\}$ and $dim\ M_2 = 2$. Since $\{\bar{x}_6, \bar{x}_7\}$ spans M_2, put $A_2 = [\bar{x}_6, \bar{x}_7] = \dfrac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Then $E_2 = A_2^{-1} = 2\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ and we require $R_2 \geq 7$ such that $\|z_n\|_\infty \leq 1/2$ for all $n > R_2$, where $z_n = -E_2\bar{x}_n$. We find that

$R_2 = 7$ and put $B_2 = [\bar{x}_8] = [1/4, 0]'$. Then $C_2 = -E_2 B_2 = [1/2, 1/2]'$ and rows 6 to 8 of P are

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}'$$

Continuing this process yields $P' =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -\frac{1}{2} & 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 1 & \frac{1}{8} & \frac{1}{8} & \cdots \end{bmatrix}$$

We now construct R. Using (8), since $-\bar{x}_1 = \bar{x}_2 + \bar{x}_3$ and $-\bar{x}_2 = \bar{x}_1 + \bar{x}_3$, columns 1 and 2 are both $\delta_1 + \delta_2 + \delta_3$. Using (9) it is evident that column 3 is also $\delta_1 + \delta_2 + \delta_3$. For every $i > l = 3$, (10) gives $v_i = -\bar{x}_{t+i} = -\bar{x}_{2+i}$ and the ith column of R is $-\bar{x}_{2+i,1}\delta_1 - \bar{x}_{2+i,2}\delta_2 + \delta_{2+i}$. Letting (ϕ_n) be given by $\phi_1 = \phi_2 = 1$ and $\phi_{3k} = \phi_{3k+1} = \phi_{3k+2} = 1/2^{k-1}$ for all $k \geq 1$, we obtain $Q = \left[\; P \mid S \;\right] =$

$$\left[\begin{array}{ccccccccccccccccc}
1 & -\frac{1}{2} & 1 & 1 & 1 & 0 & \frac{1}{2} & -\frac{1}{8} & 0 & \frac{1}{8} & -\frac{1}{32} & 0 & \frac{1}{32} & -\frac{1}{128} & 0 & \frac{1}{128} & \cdots \\
1 & 0 & 1 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{8} & \frac{1}{8} & 0 & -\frac{1}{32} & \frac{1}{32} & 0 & -\frac{1}{128} & \frac{1}{128} & \cdots \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots & & & & & & & & & \\
1 & 0 & 0 & \cdots & & & & & & & & & & & & & \\
1 & 1 & 0 & \cdots & & & & & & & & & & & & & \\
1 & \frac{1}{2} & 0 & \cdots & 0 & 1 & 0 & \cdots & & & & & & & & & \\
1 & \frac{1}{2} & 0 & & \cdots & 0 & 1 & 0 & \cdots & & & & & & & & \\
1 & 1 & 0 & & & \cdots & 0 & \frac{1}{2} & 0 & \cdots & & & & & & & \\
1 & \frac{1}{4} & 0 & & & & \cdots & 0 & \frac{1}{2} & 0 & \cdots & & & & & & \\
1 & \frac{1}{4} & 0 & & & & & \cdots & 0 & \frac{1}{2} & 0 & \cdots & & & & & \\
1 & 0 & 0 & & & & & & \cdots & 0 & \frac{1}{4} & 0 & \cdots & & & & \\
\vdots & & & & & & & & & & & \ddots & & & & &
\end{array}\right]$$

The rows of Q form the compact set of extreme points, X, of the α-polytope $K_2 = \bar{co}\ X$.

5 Concluding Remarks

(i) The polytope K of Example 2 in [5] is the closed convex hull of an extreme set X with the following structure: if K' is the closed convex hull of $X\backslash\{x_1, x_2\}$, then K' lies in a hyperplane H, and x_1 and x_2 are the end points of a line segment which intersects the interior of K' in exactly one point. If we apply the construction method of this paper to $\bar{X}$, we obtain an α-polytope which exhibits this structure.

(ii) The authors consider this paper to leave some interesting questions unanswered. In finite dimensions, an equivalence of Gale transforms has been defined which corresponds to the combinatorial equivalence of the polytopes with these transforms. Can such an equivalence of transforms of infinite dimensional α-polytopes be defined, and can it be used to determine an equivalence of α-polytopes?

The construction of §3 is capable of yielding topologically distinct α-polytopes. What do polytopes constructed from the same Gale transform have in common? Finally, are there alternative Gale basis construction methods, and if so, how would α-polytopes so constructed differ from those constructed here?

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