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Optical Tomography for Variable Refractive Index with Angularly Averaged Measurements

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In optical tomography one seeks to use near-infrared light to determine the optical absorption and scattering properties of a medium $X \subset \mathbb{R}^n$. If the refractive index is constant throughout the medium, the steady-state case is modeled by the stationary linear transport equation in terms of the Euclidean metric. In this work we consider the case of variable refractive index where the dynamics are modeled by writing the transport equation in terms of a Riemannian metric; in the absence of interaction, photons follow the geodesics of this metric. In particular we study the problem where our measurements allow the application of an in-going flux depending on both position and direction, but we allow only a weighted average measurement of the out-going flux. We show that making measurements on all of ∂X determines the extinction coefficient and that once this is known, under additional assumptions, measurements at a single point on ∂X determine the scattering kernel.

Keywords Boltzmann equation; Integral geometry; Optical tomography; Riemannian metric; Transport equation.

Mathematics Subject Classification Primary 35R35; Secondary 35Q99.

1. Introduction

Linear transport equations model, among other things, the propagation of the energy density of waves in heterogeneous media [2, 5, 19, 24], neutrons in nuclear reactors [15], and near-infrared photons in tissue. Recently, photon propagation has been applied in optical tomography for use in medical imaging [1, 18]. In the absence of scattering, propagation is usually assumed to be along straight lines, as would be the case for light propagation in a medium with constant index of refraction. This amounts to writing the transport equation in terms of the Euclidean metric. Here we consider the situation where, in the absence of scattering, propagation is along

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geodesics of a Riemannian metric. We consider this a model for optical tomography in a medium with continuously varying refractive index. When prescribed in-going, and measured out-going, fluxes are allowed to depend on both position and direction (in other words, measurements are fully "angularly resolved"), the problem of optical tomography in the Euclidean setting has been well studied; the results most closely related to the present article are those in [6], [7] and [22]. The Riemannian case is considered in [13] and [14]. Such measurements are, however, unrealistic since angular resolution of out-going flux is difficult to achieve. In [12], Langmore introduces an angularly averaged measurement operator $\mathcal{M}$ and proves that angularly averaged measurements taken at every point on the boundary of the medium enables reconstruction of the extinction coefficient σ; once σ is known, measurements made at a single boundary point allow unique determination of the scattering kernel under some additional assumptions. In particular, it is assumed that scattering is supported inside of X, the scattering phase function is known, and the scattering kernel is small. It is also assumed that σ and the phase function are "close to" real-analytic. Here we consider the same averaged measurement operator in the case of the transport equation in terms of a Riemannian metric and obtain in this setting results analogous to those of [12].

Let $X \subset \mathbb{R}^n$ be an open bounded set with smooth boundary ∂X and let g be a smooth Riemannian metric on $\overline{X}$. We shall make geometric assumptions on the Riemannian manifold (X, g) in due course. If $u(x, v)$ represents the density of particles at position x with velocity v in the unit tangent sphere at x, $\Omega_x X$, then the stationary linear transport equation is

$$Tu(x, v) = -\mathcal{D}u(x, v) - \sigma(x, v)u(x, v) + \int_{\Omega_x X} k(x, v', v)u(x, v')dv' = 0. \qquad (1)$$

The operator $\mathcal{D}$ is the derivative along the geodesic flow (see (2) below) which in the case of g being Euclidean is simply $\mathcal{D}u(x, v) = v \cdot \nabla_x u(x, v)$. The measure dv' is the volume form on $\Omega_x X$ induced from the Euclidean volume form on $T_x X$ determined by g at x; here $T_x X$ is the full tangent space to X at x.

The case of a Euclidean metric corresponds to transport in material with a constant index of refraction. Though this is easier to study, in practice it is always an approximation. For example, in [3] a transport equation is derived as a limiting case of Maxwell's equations with non-constant (yet isotropic) permeability. The result is an inhomogeneous wave-speed $c = c(x)$, and hence $c/c_0 := n = n(x)$ is variable as well. Here c_0 is the speed of light in a vacuum, and n is the refractive index. Examination of this result shows that our model (1) with conformally Euclidean metric $g_{ij}(x) = c_0^{-2}n(x)^2\delta_{ij}$ correctly describes energy density propagation in this isotropic material. This is seen by noting that (in [3]) the solution to the transport equation follows (in the absence of scattering) bicharacteristics of the Hamiltonian $\omega(x, k) = c(x)|k|$. Here, the wave-number k is the dual variable to x. As is well known, the Hamiltonian $H(x, k) := \omega(x, k)^2 = c(x)^2|k|^2$ defines identical trajectories. The physical space projections, $x(t)$, are then geodesics of the metric $g_{ij} = c^2\delta_{ij}$, and travel with speed $n(x)/c_0 = c(x)$. See e.g. [23]. We thus arrive at $g_{ij} = c_0^{-2}n^2\delta_{ij}$ as the proper metric for transport in this case. It is not possible to state a general correspondence between refractive index in wave equations and metrics in limiting transport equations. For example, transport limits for dispersive wave equations can lead to trajectories turning around and re-tracing their path. We also mention that the model (1) appears in this form in [21].

Define the "incoming" and "outgoing" bundles

$$\Gamma_\pm = \{(x, v) : x \in \partial X, \text{ and } \pm \langle v, v_x \rangle > 0\}$$

where v_x is the unit outer normal vector to the boundary ∂X at x and $\langle \cdot, \cdot \rangle$ is the inner product, each with respect to g at x.

Given an incoming flux of particles u_- defined on Γ_-, let u be the solution, should it exist, to $Tu = 0$ with the boundary condition $u|_{\Gamma_-} = u_-$. The albedo operator is defined to be $\mathscr{A} : u_- \mapsto u|_{\Gamma_+}$. Determination of σ and k from knowledge of $\mathscr{A}$ is the subject of [4, 7, 22, 23] in the Euclidean setting, and of [13] and [14] in the Riemannian setting. Here we consider the problem with less information. Instead of knowing the angularly resolved measurement $u(x, v)$ on Γ_+ we will assume only knowledge of an *average* over outgoing directions of u at $x \in \partial X$. To be more precise, for $x \in \partial X$, define

$$\Omega_x^\pm X = \{v \in \Omega_x X : \pm\langle v, v_x \rangle > 0\}$$

(and so $\Gamma_\pm$ are the disjoint unions of the $\Omega_x^\pm X$ over $x \in \partial X$). We will also use the somewhat unconventional notation

$$\Omega^2 X := \{(x, v, w) : x \in X, v, w \in \Omega_x X\}.$$

We shall denote by T^{-1} the solution operator to the boundary value problem $Tu = 0$, $u|_{\Gamma_-} = u_-$, that is, $u = T^{-1}u_-$.

Definition 1.1. The data of which we will be assuming knowledge consists of *angularly averaged* measurements on ∂X, weighted with respect to a prescribed function $m(x, v)$. Specifically, given u_- on Γ_- and $u = T^{-1}u_-$ we define $\mathscr{M} : L^1(\Gamma_-, d\mu) \to L^1(\partial X)$ by (the mapping properties will be verified later)

$$\mathscr{M}u_-(x) := \int_{\Omega_x^+ X} u(x, v) m(x, v) dv.$$

We require that $m \in L^\infty(\Gamma_+)$, that m does not vanish on Γ_+, and that $|m(x, v)/\langle v, v_x \rangle|$ be bounded on Γ_+.

The function m corresponds to the limitations of the measurement apparatus. It may represent a limited aperture or, for example, when $m(x, v) = \langle v, v_x \rangle$, the measurement is power flux exiting the boundary.

When the ambient metric is Euclidean, this problem was introduced and studied in [12] where an analysis of a singular decomposition of the Schwarz kernel of $\mathscr{M}$ (into an infinite series) is performed. From the most singular term, corresponding to the ballistic particles which do not scatter, the extinction coefficient $\sigma(x)$ is recovered. To recover k, additional assumptions (on σ, m and k) are made that allow one to view the term linear in k as a weighted X-ray transform and apply results from [10] to show injectivity of this map. Under a smallness assumption on k, injectivity of the full nonlinear map (as well as stability) is then proved. The precise formulation is very similar to what will be presented in the present article where we obtain analogous results when the background metric is Riemannian.

Here we state our basic assumptions on the coefficients in (1). We assume σ depends only on position. This slightly restrictive (but physically realistic) assumption on σ is necessary: If $k \equiv 0$, and g is Euclidean, then this problem reduces to the usual X-ray transform. One can then show that if σ depends on the direction v (as opposed to $|v|$), no boundary measurement can uniquely determine σ (see the introduction of [7]). The "monochromatic" assumption $|v|_g = 1$ greatly simplifies the problem, and is commonly used. We will also assume that $0 \leq \sigma \in L^\infty(X)$, $k \in L^\infty(\Omega^2 X)$ are bounded functions. Solution to the forward problem also requires some sort of bound on energy production (and therefore k), which is sometimes given in relation to σ, [8, 20]. Here we assume $\|k\|_{L^\infty} < (|\mathbb{S}^{n-1}|\text{diam}(X))^{-1}$. Unique recovery of the scattering kernel requires the assumption that it takes the form $k(x)\Theta(x, v', v)$ for some known function Θ (this is not assumed for recovery of σ). This assumption is reasonable (and quite common) and Θ can model the known underlying scattering process, whereas k models the density of scattering objects. Both of our main results require additional smallness assumptions on k. Our uniqueness theorem for k requires that m, σ, Θ and g are all close to real analytic. These last two assumptions will be made precise in the statements of Theorems 2.1 and 2.2.

Remark. The scattering term $k(x, v', v)$ in (1) is proportional to the probability of a particle with position x and velocity $v' \in \Omega_x X$ being scattered to velocity $v \in \Omega_x X$. The total extinction term $\sigma(x)u(x, v)$ accounts for loss of energy in direction v due to absorption at position x as well as scattering into other directions. The latter is given by $u(x, v) \int k(x, v, v'')dv''$. Since σ and k are separated mathematically, we treat them as independent functions in this paper, but keep in mind the following caveat: In the physical setup, assumptions made about σ imply assumptions about k and vice versa.

We must also place restrictions on the geometry of (X, g). Recovery of both σ and k relies on injectivity of the (weighted) geodesic X-ray transform. It is for this reason that we assume the metric is "simple":

Assumption 1.1. (X, g) is simple: X is strictly convex, and for any $x \in \overline{X}$ the exponential map $\exp_x : \exp_x^{-1}(\overline{X}) \to \overline{X}$ is a diffeomorphism (and consequently X is diffeomorphic to a ball).

To state the precise results it is necessary to introduce the volume forms on $\Omega_x X$ and $\Gamma_\pm$. On X we have the naturally defined volume form of the metric. At any $x \in X$, the volume form dv on $\Omega_x X$ is the form induced from the Euclidean volume on $T_x X$ defined by the metric g at x. The resulting form on ΩX is the Liouville form and is preserved under the geodesic flow of g. We denote by $d\mu$ the induced volume form on $\Gamma_\pm$ which has the property that $dt\, d\mu(x', v')$ is the pull-back of the Liouville form by the geodesic flow. Equivalently, we have the induced volume form of ∂X included in $\overline{X}$; if x' are local coordinates for ∂X and dx' is this volume form on ∂X, then it holds that

$$d\mu(x', v') = |\langle v', \nu_{x'} \rangle| dv'\, dx'.$$

When not explicitly stated otherwise, measures on (for example) ∂X and X will be the volume forms defined by the metric g. Thus we shall write, for example, $L^1(\partial X)$

and $L^1(X)$ without further reference to the measures on these spaces. Similarly, for example, $L^1(\Omega X)$ is with respect to the Liouville form, and $L^1(\Gamma_\pm)$ is with respect to $d\mu$.

The rest of the paper is organized as follows. In Section 2 we give our main results on the unique determination of total cross section σ and scattering k. In Section 3 our notation for the forward problem is introduced. In Section 4, standard results for the albedo operator are recounted, and new results for the averaged albedo operator are obtained as a result. Here we also see how geometric considerations involving curvature play a role. In Section 5 a proof of the unique determination of the extinction coefficient is given. In Section 6 the same is done for the scattering kernel.

2. Statement of the Main Results

If $\mathcal{H} \subset \Gamma_-$, then by $\Gamma(\mathcal{H})$ we mean the set of geodesics with initial data in $\mathcal{H}$:

$$\Gamma(\mathcal{H}) = \{\gamma_{(x',v')}(t) : (x', v') \in \mathcal{H},\ 0 \le t \le \tau_+(x', v')\}.$$

In Lemma 4.2 we define two constants C_{κ_m} and C_{κ_M} which reflect bounds on the possible size of Jacobi fields.

Theorem 2.1 (Recovery of σ). *Suppose that*

$$\|k\|_{L^\infty} < \min\left\{[(C_{\kappa_m} C_{\kappa_M})^{n-1} \operatorname{diam} X |\mathbb{S}^{n-1}|]^{-1}, [\operatorname{diam} X |\mathbb{S}^{n-1}|]^{-1}\right\},$$

and that $\mathcal{H}_\sigma \subset \Gamma_-$ is open and such that the geodesic X-ray transform restricted to $\Gamma(\mathcal{H}_\sigma)$ is injective. Suppose that $\sigma = \sigma(x)$ depends on position only. Then σ is uniquely determined by $\{(u_-, \mathcal{M}u_-) : u_- \in L^1(\mathcal{H}_\sigma)\}$.

Fixing $D > 0$ and making the definition

$$\mathcal{K}_\varepsilon^D := \{k \in L^\infty(X) : \operatorname{dist}(\operatorname{supp}(k), \partial X) > D, \|k\|_{L^\infty} \le \varepsilon\}$$

we also have the following uniqueness result for k.

Theorem 2.2 (Recovery of k). *Suppose the scattering kernel has the form $k(x)\Theta(x, v', v)$, with (g, m, σ, Θ) known and real analytic, and with both m and Θ non-vanishing. Let $\mathcal{H}_k \subset \Gamma_-$ be open, and assume that for every $(x, v) \in TX\backslash\{0\}$, there exists $\gamma \in \Gamma(\mathcal{H}_k)$ through x and normal to v at x. Then there exits ε sufficiently small such that for* a.e. *$x \in \partial X$, knowledge of $\{(u_-, \mathcal{M}u_-(x)) : u_- \in L^1(\mathcal{H}_k)\}$ uniquely determines k within the class $\mathcal{K}_\varepsilon^D$.*

Furthermore, ε may be chosen such that this result holds in some C^2 neighborhood of (m, σ), and some C^{3n} neighborhood of (Θ, g).

Remark. For Theorem 2.2 to hold, we at least require $\varepsilon < \left[\|\Theta\|_{L^\infty} \operatorname{diam} X |\mathbb{S}^{n-1}|\right]^{-1}$. As mentioned earlier, the Schwarz kernel term linear in k is a weighted X-ray transform. To "invert" the transform, one uses the open mapping theorem. This results in a constant (and therefore a smallness requirement on ε) that cannot be calculated explicitly. We take ε small enough to meet both of these requirements.

Since real analytic functions are dense, we have a uniqueness result for an open and dense set of (m, g, σ, Θ), where it is still required that g be a simple metric.

3. The Forward Problem

Given $(x, v) \in \Omega X$ we denote by $\gamma_{(x,v)}(\cdot)$ the geodesic uniquely determined by $\gamma_{(x,v)}(0) = x$, $\dot{\gamma}_{(x,v)}(0) = v$. We will use the shorthand notation

$$\vec{\gamma}_{(x,v)}(t) := (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)).$$

Define the "distance to boundary" functions

$$\tau_{\pm}(x, v) = \min\{t \geq 0 : \gamma_{(x,v)}(\pm t) \in \partial X\}.$$

Since (X, g) is simple, these functions are well-defined and finite. The operator $\mathscr{D}$ in (1) is the derivative along the geodesic flow and is defined by

$$\mathscr{D}u(x, v) = \frac{\partial}{\partial t}\bigg|_{t=0} u(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)). \tag{2}$$

If $(x^i, y^i)_{i=1}^n$ are local coordinates for ΩX with the (y^i) with respect to the natural basis $\left(\frac{\partial}{\partial x^i}\right)$ then in these coordinates

$$\mathscr{D}f = \frac{\partial f}{\partial x^i} y^i + \frac{\partial f}{\partial y^i}(-y^j y^k \Gamma^i_{jk})$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of g.

Given two points $x, y \in X$ there is a unique geodesic from x to y; let $d(x, y) = d_g(x, y)$ be the geodesic distance between x and y and let $v(x, y)$ be the tangent vector to this geodesic at x. Define

$$E(x, y) := \exp\left\{-\int_0^{d(x,y)} \sigma\big(\gamma_{(x,v(x,y))}(t)\big)dt\right\}.$$

Note, using the fact $\gamma_{(y,v(y,x))}(d(y, x) - s) = \gamma_{(x,v(x,y))}(s)$ we have

$$E(x, y) = E(y, x).$$

We define the following operators which arise naturally from (1). When $k \equiv 0$, the solution operator to $Tu = 0$, $u\big|_{\Gamma_-} = u_-$ is

$$Ju_-(x, v) := E\big(x, \gamma_{(x,v)}(-\tau_-(x, v))\big)u_-\big(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))\big).$$

Define

$$T_1 f(x, v) := \int_{\Omega_x X} k(x, v', v) f(x, v') dv'$$

and

$$Kf(x, v) := \int_0^{\tau_-(x,v)} E\big(x, \gamma_{(x,v)}(t - \tau_-(x, v))\big) T_1 f\big(\vec{\gamma}_{(x,v)}(t - \tau_-(x, v))\big) dt$$

$$= \int_0^{\tau_-(x,v)} E\big(x, \gamma_{(x,v)}(t - \tau_-(x,v))\big)$$
$$\times \int_{\Omega_{\gamma_{(x,v)}(t-\tau_-(x,v))}X} k\big(\gamma_{(x,v)}(t-\tau_-(x,v)), v', \dot{\gamma}_{(x,v)}(t-\tau_-(x,v))\big)$$
$$\times f(\gamma_{(x,v)}(t-\tau_-(x,v)), v')dv'\,dt.$$

The solution to the boundary value problem $Tu = 0$, $u\big|_{\Gamma_-} = u_-$ then satisfies the integral equation

$$(I - K)u = Ju_-. \tag{3}$$

4. The Averaged Albedo Operator

The goal of this section is to establish Theorem 4.1 where we show that the averaged albedo operator $\mathcal{M}$ can be expressed as the infinite sum of the operators $\mathcal{M}_i$ (see Definition 4.1), and where we give expressions for the Schwartz kernel of $\mathcal{M}_i$, $i \geq 1$. We first present a series expansion for the albedo operator $\mathcal{A}$ (see Lemma 4.1) for boundary fluxes $u_- \in L^1(\Gamma_-, d\mu)$ which is developed in [13]. The definition of the measure on Γ_- makes immediate the following lemma which facilitates a useful and repeatedly used change of variables in integration.

Lemma 4.1 (Change of Variables). *If $u \in L^1(\Omega X)$ then*

$$\int_X \int_{\Omega_x X} u(x,v)dv_x\,dx = \int_{\Gamma_\pm} \int_0^{\tau_\mp(x',v')} u(\gamma_{(x',v')}(t), \dot{\gamma}_{(x',v')}(t))dt\,d\mu(x',v').$$

For $(x,v) \in \Omega X$ define $\tau(x,v) := \tau_-(x,v) + \tau_+(x,v)$. It is shown in [13] that if $\|k\|_{L^\infty(\Omega^2 X)} < (|\mathbb{S}^{n-1}|\operatorname{diam} X)^{-1}$ then $I - K$ is invertible on $L^1(\Omega X, \tau^{-1}dv\,dx)$, and so (3) is invertible on this space. The following proposition is also proven in [13].

Proposition 4.1. *Let k satisfy $\|k\|_{L^\infty(\Omega^2 X)} < \big(|\mathbb{S}^{n-1}|\operatorname{diam} X\big)^{-1}$. The albedo operator $\mathcal{A} : L^1(\Gamma_-, d\mu) \to L^1(\Gamma_+, d\mu)$ is continuous and has the expansion*

$$\mathcal{A}u_-(x,v) := (T^{-1}u_-)\big|_{\Gamma_+} = \sum_{i=0}^{\infty} K^i J u_-(x,v)\Big|_{\Gamma_+},$$

the series converging in $L^1(\Gamma_+, d\mu)$.

In order to simplify the presentation of what follows, we introduce some notation. If $x_1, \ldots, x_j$ are points in X, then

$$E(x_1, x_2, \ldots, x_j) := E(x_1,x_2)E(x_2,x_3)\ldots E(x_{j-1},x_j) = \prod_{i=1}^{j-1} E(x_i, x_{i+1}). \tag{4}$$

If $y \in X$, consider $z = z_y(t,v) = \gamma_{(y,v)}(t)$ defined from the "polar" coordinates $(t,v) \in \mathbb{R} \times \Omega_y X$. Let $J_y(z)$ denote the Jacobian determinant $|\det \partial z/\partial(t,v)|^{-1}$ of this change of variables. For a given z, let (t,v) be its polar coordinate expression, and let $\{v^1, \ldots, v^n\}$ be an orthonormal basis for $T_y X$ with $v^1 = v$. Let $Y_{y,z,i}$ be the Jacobi field along $\gamma_{(y,v)}(\cdot)$ with initial data $Y_{y,z,i}(0) = 0$, $\dot{Y}_{y,z,i} = v^i$, $2 \le i \le n$. Then $J_y(z)$ is given by the expression (see [17])

$$J_y(z) = \prod_{i=2}^{n} |Y_{y,z,i}(d(y,z))|^{-1}. \tag{5}$$

The notation is chosen to remind the reader that J_y comes from geodesic polar coordinates based at y. We will frequently need products of such change of volume elements and so introduce the notation

$$\mathcal{J}(y_1, \ldots, y_j) := \prod_{i=2}^{j} J_{y_i}(y_{i-1}). \tag{6}$$

We point out that in the case of g being Euclidean, J_y and $\mathcal{J}$ take the simple forms

$$J_y(z) = \frac{1}{|y-z|^{n-1}}, \quad \text{and} \quad \mathcal{J}(y_1, \ldots, y_j) = \prod_{i=2}^{j} \frac{1}{|y_i - y_{i-1}|^{n-1}}.$$

Lemma 4.2. *With $J_y(z)$ defined as above, there exists a constant C_{κ_M}, such that for all $y, z \in X$*

$$J_y(z) \le \frac{C_{\kappa_M}}{d(y,z)^{n-1}}. \tag{7}$$

Further, there exists a constant C_{κ_m} such that if Y is any Jacobi field along a geodesic $\gamma(t)$ with $Y(0) = 0$ and $\dot{Y}(0) \in \dot{\gamma}^{\perp}(0) \subset \Omega_{\gamma(0)}X$, then

$$\frac{|Y(t)|}{t} \le C_{\kappa_m} \tag{8}$$

for all $0 \le t \le \tau_+(\gamma(0), \dot{\gamma}(0))$.

Remark. As the Euclidean expression for J_y would suggest, $J_y(x)$ has a (weak) singularity for x near y. There is another sigularity that could possibly occur due to curvature. Specifically, geodesics starting at y can "focus" near a point x. This means that the integral over directions carries a strong influence from a neighborhood of this point. Simplicity of (X, g) however ensures that this focus never causes a singularity. The notation C_{κ_M} is suggestive of the fact that the size of this constant depends on the maximal sectional curvature of (X, g).

Proof. Let Y_i be a Jacobi field of the form described in the second part of the lemma. Let κ_M be an upper bound for the sectional curvature of (X, g), the existence of which is guaranteed by compactness of $\overline{X}$. Then Rauch comparison gives

$$|Y_i(d(y,z))| \ge \begin{cases} \frac{1}{\sqrt{\kappa_M}} \sin(\sqrt{\kappa_M}\, d(y,z)), & \kappa_M > 0, \\ d(y,z), & \kappa_M = 0, \\ \frac{1}{\sqrt{-\kappa_M}} \sinh(\sqrt{-\kappa_M}\, d(y,z)), & \kappa_M < 0. \end{cases}$$

First suppose that $\kappa_M > 0$; if $d(y,z) \leq \pi/(2\sqrt{\kappa_M})$ then $\sin(\sqrt{\kappa_M} d) \geq 2\sqrt{\kappa_M} d/\pi$ and so

$$|Y_i(d(y,z))| \geq \frac{2}{\pi} d(y,z).$$

Simplicity of (X,g) guarantees the absence of conjugate points and so Jacobi fields $Y_i(d)$ of the form considered here cannot vanish for $d > 0$, and in particular for $d \geq \pi/(2\sqrt{\kappa_M})$, and so by compactness $|Y_i(d(y,z))| \geq Cd(y,z)$. If $\kappa_M < 0$ then $\sinh(\sqrt{-\kappa_M} d(y,z)) > \sqrt{-\kappa_M} d(y,z)$. Given the expression (5) for $J_y(z)$, these estimates yield (7).

For (8), let κ_m be a lower bound for the sectional curvature of (X,g). Theorem 4.5.2 of [11] gives

$$|Y(t)| \leq \begin{cases} \dfrac{1}{\sqrt{\kappa_m}} \sin(\sqrt{\kappa_m} t), & \kappa_m > 0, \\ t, & \kappa_m = 0, \\ \dfrac{1}{\sqrt{-\kappa_m}} \sinh(\sqrt{-\kappa_m} t), & \kappa_m < 0 \end{cases}$$

which, as above, in turn implies (8). □

Definition 4.1. For each $i \geq 0$ we define the operator $\mathcal{M}_i$ by

$$\mathcal{M}_i u_-(x) := \int_{\Omega_x^+ X} K^i J u_-(x,v) m(x,v) dv.$$

Theorem 4.1. *Let* $\|k\|_{L^\infty(\Omega^2 X)} < \left(|\mathbb{S}^{n-1}| \operatorname{diam} X\right)^{-1}$. *Then* $\mathcal{M}_i, \mathcal{M} : L^1(\Gamma_-, d\mu) \to L^1(\partial X)$ *continuously, and for almost every* $x \in \partial X$, $\mathcal{M}$ *has the expansion*

$$\mathcal{M} u_-(x) = \sum_{j=0}^{\infty} \mathcal{M}_j u_-(x) = \sum_{j=0}^{\infty} \int_{\Gamma_-} \alpha_j(x,x',v') u_-(x',v') d\mu(x',v')$$

where $\alpha_0(x,\cdot,\cdot)$ *is a distribution supported on a manifold of dimension* $n-1$, *and for* $j \geq 1$, $\alpha_j(x,x',v')$ *are the following Schwartz kernels*: *when* $j = 1$,

$$\alpha_1(x,x',v') = \int_0^{\tau_+(x',v')} k(\vec{\gamma}_{(x',v')}(t), \bar{w}_1) E(x', \gamma_{(x',v')}(t), x) m(x, \hat{w}_1) J_x(\gamma_{(x',v')}(t)) dt \quad (9)$$

where $\bar{w}_1, \hat{w}_1$ *are the initial and final tangent vectors of the geodesic from* $\gamma_{(x',v')}(t)$ *to* x, *and* E *and* J_x *are defined in* (4), *and* (5); *for* $j \geq 2$,

$$\alpha_j(x,x',v') = \int_0^{\tau_+(x',v')} \int_X \cdots \int_X k(\vec{\gamma}_{(x',v')}(t), \bar{w}_1) \prod_{i=2}^{j} k(y_i, \hat{w}_{i-1}, \bar{w}_i) \times E(x', \gamma_{(x',v')}(t), y_2, \ldots, y_j, x) \mathcal{J} m(x, \hat{w}_j) dy_j \ldots dy_2 \, dt \quad (10)$$

where:

- $\bar{w}_1$, $\hat{w}_1$ *are the initial and final tangent vectors of the geodesic joining* $\gamma_{(x',v')}(t)$ *to* y_2,

- $\bar{w}_i$, $\hat{w}_i$ *are the initial and final tangent vectors of the geodesic joining* y_i *to* y_{i+1} *for* $i = 2, \ldots, j-1$,
- $\bar{w}_j$, $\hat{w}_j$ *are the initial and final tangent vectors of the geodesic joining* y_j *to* x, *and*
- $\mathcal{J} = \mathcal{J}(\gamma_{(x',v')}(t), y_2, \ldots, y_j, x)$.

Proof. First suppose u_- is non-negative. Then since K and J preserve non-negativity,

$$\|\mathcal{M}_i u_-\|_{L^1(\partial X)} \leq \|\mathcal{M} u_-\|_{L^1(\partial X)} \leq \int_{\partial X}\int_{\Omega_x^+ X} |\mathcal{A}u_-(x,v)| \frac{|m(x,v)|}{|\langle v, \nu_x\rangle|} |\langle v, \nu_x\rangle| dv\, dx$$
$$\leq \left\| \frac{m(x,v)}{|\langle v, \nu_x\rangle|}\right\|_{L^\infty(\Gamma_+)} \|\mathcal{A}u_-\|_{L^1(\Gamma_+, d\mu)}.$$

We obtain the same bound on general u_- by decomposing $u_- = u_-^+ - u_-^-$. So by Proposition 4.1, $\mathcal{M}_i, \mathcal{M} : L^1(\Gamma_-, d\mu) \to L^1(\partial X)$ continuously.

Next by Lemma 4.1,

$$\mathcal{M}u_-(x) := \int_{\Omega_x^+ X} \mathcal{A}u_-(x,v) m(x,v) dv = \int_{\Omega_x^+ X} \sum_{j=0}^{\infty} K^j J u_-(x,v) m(x,v) dv.$$

Now

$$\left\| \sum_{j=0}^{\infty} K^j J u_- m \right\|_{L^1(\Gamma_+, dv_x\, dx)} = \left\| \sum_{j=0}^{\infty} K^j J u_- \frac{m}{|\langle v, \nu_x\rangle|}\right\|_{L^1(\Gamma_+, d\mu)}$$
$$\leq \left\| \frac{m(x,v)}{|\langle v, \nu_x\rangle|}\right\|_{L^\infty(\Gamma_+)} \|\mathcal{A}u_-\|_{L^1(\Gamma_+, d\mu)}$$

and so for almost every $x \in \partial X$, $\sum_{j=0}^{\infty} K^j J u_-(x,\cdot) m(x,\cdot) \in L^1(\Omega_x^+ X)$. Again, since K and J preserve non-negativity, we may use monotone convergence to interchange the integration with the summation in the above to obtain

$$\mathcal{M}u_-(x) = \sum_{j=0}^{\infty} \int_{\Omega_x^+ X} K^j J u_-(x,v) m(x,v) dv = \sum_{j=0}^{\infty} \mathcal{M}_j u_-(x).$$

For each j, let $\alpha_j(x, x', v') \in \mathcal{D}'(\partial X \times \Gamma_-)$ be the Schwarz kernel of $\mathcal{M}_j$. When $j = 0$, for fixed $x \in \partial X$, $\alpha_0(x, x', v')$ is a singular distribution. To see this, let $F = \{(x', v') \in \Gamma_- : \gamma_{(x',v')}\tau_+(x', v') = x\}$, let $U \subset F$ be open in F, and let u_n be a sequence of smooth functions on Γ_- such that $u_n\big|_U = 1$ and $u_n \to \chi_U$ as $n \to \infty$ (in C^0 norm), where χ_U is the characteristic function of U. Note that F is an $n-1$ dimensional sub-manifold lying in the $2n-2$ dimensional Γ_-. Then

$$\begin{aligned}\int_{\Gamma_-} \alpha_0(x, x', v') u_n(x', v') d\mu &= \int_{\Omega_x^+ X} J u_n(x,v) m(x,v) dv \\ &= \int_{\Omega_x^+ X} E\big(x, \gamma_{(x,v)}(-\tau_-(x,v))\big) u_n\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big) m(x,v) dv \\ &\geq \int_{\{v \in \Omega_x^+ X : \vec{\gamma}_{(x,v)}(-\tau_-(x,v)) \in U\}} E\big(x, \gamma_{(x,v)}(-\tau_-(x,v))\big) m(x,v) dv \\ &\neq 0\end{aligned}$$

for every n. Now $u_n \to 0$ in every $L^p(\Gamma_-)$, $1 \le p \le \infty$ and so $\alpha_0 \notin L^p(\Gamma_-)$ for every such p. The above also shows that the support of $\alpha_0(x,\cdot,\cdot)$ is at least F; further, if $\varphi \in C_c^\infty(\Gamma_-)$ is such that $\operatorname{supp}\varphi \cap F = \emptyset$ then, as above,

$$\int_{\Gamma_-} \alpha_0(x,x',v')\varphi(x',v')d\mu = 0$$

and so $\operatorname{supp}\alpha_0(x,\cdot,\cdot) = F$.

We now derive the expressions (9) and (10). When $j=1$, let ϕ_- be a function on Γ_-. We have

$$\begin{aligned}
\mathscr{M}_1\phi_-(x) &= \int_{\Omega_x^+ X} KJ\phi_-(x,v)dv \\
&= \int_{\Omega_x^+ X}\int_0^{\tau_-(x,v)} E\big(x,\gamma_{(x,v)}(t-\tau_-(x,v))\big)T_1J\phi_- \\
&\quad \times (\vec{\gamma}_{(x,v)}(t-\tau_-(x,v))dt\, m(x,v)dv \\
&= \int_{\Omega_x^+ X}\int_0^{\tau_-(x,v)} E\big(x,\gamma_{(x,v)}(t-\tau_-(x,v))\big) \\
&\quad \times \int_{\Omega_y X} k(y,\hat{w},\bar{w}_1)E\big(\gamma_{(x,v)}(t-\tau_-(x,v)),\gamma_{(y,\hat{w})}(-\tau_-(y,\hat{w}))\big) \\
&\quad \times \phi_-(\vec{\gamma}_{(y,\hat{w})}(-\tau_-(y,\hat{w}))d\hat{w}\,dt\,m(x,v)dv
\end{aligned}$$

where $(y,\bar{w}_1) = \vec{\gamma}_{(x,v)}(t-\tau_-(x,v))$

$$\begin{aligned}
&= \int_X\int_{\Omega_y X} E\big(x,y,\gamma_{(y,\hat{w})}(-\tau_-(y,\hat{w}))\big)k(y,\hat{w},\bar{w}_1)\phi_-\big(\vec{\gamma}_{(y,\hat{w})}(-\tau_-(y,\hat{w}))\big)d\hat{w} \\
&\quad \times m(x,\hat{w}_1)J_x(y)dy
\end{aligned}$$

where $\bar{w}_1$, $\hat{w}_1$ are the initial and final tangent vectors (respectively) of the geodesic from y to x,

$$\begin{aligned}
&= \int_{\Gamma_-}\int_0^{\tau_+(x',v')} k(\vec{\gamma}_{(x',v')}(t),\bar{w}_1)E(x',\gamma_{(x',v')}(t),x)m(x,\hat{w}_1)J_x(\gamma_{(x',v')}(t))dt \\
&\quad \times \phi_-(x',v')d\mu(x',v').
\end{aligned}$$

This proves (9).

For brevity of exposition, we shall cease to write out the arguments of the functions E writing simply $E(\cdot)$ (recall also the convention (4)). Toward deriving the result for $j=2$, we first compute

$$\begin{aligned}
&T_1KJ\phi_-(y_2,\bar{w}_2) \\
&\quad = \int_{\Omega_{y_2}X} k(y_2,\hat{w}_1,\bar{w}_2)\int_0^{\tau_-(y_2,\hat{w}_1)} E(\cdot)T_1J\phi_-\big(\vec{\gamma}_{(y_2,\hat{w}_1)}(t_1-\tau_-(y_2,\hat{w}_1))\big)dt_1\,d\hat{w}_1 \\
&\quad = \int_{\Omega_{y_2}X} k(y_2,\hat{w}_1,\bar{w}_2)\int_0^{\tau_-(y_2,\hat{w}_1)}\int_{\Omega_{y_1}X} k(y_1,\hat{w},\bar{w}_1) \\
&\qquad \times E(\cdot)\phi_-(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1,\hat{w}))d\hat{w}\,dt_1\,d\hat{w}_1
\end{aligned}$$

where $(y_1, \bar{w}_1) = \vec{\gamma}_{(y_2,\hat{w}_1)}(t_1 - \tau_-(y_2, \hat{w}_1))$

$$= \int_X \int_{\Omega_{y_1}X} E(\cdot)k(y_2, \hat{w}_1, \bar{w}_2)k(y_1, \hat{w}, \bar{w}_1)\phi_-\big(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1, \hat{w}))\big)d\hat{w}\, J_{y_2}(y_1)dy_1.$$

Now,

$$\mathcal{M}_2\phi_-(x) = \int_{\Omega_x^+ X} \int_0^{\tau_-(x,v)} E(\cdot)T_1KJ\phi_-(y_2, \bar{w}_2)dt_2\, m(x, v)dv$$

where $(y_2, \bar{w}_2) = \vec{\gamma}_{(x,v)}(t_2 - \tau_-(x, v))$,

$$\begin{aligned}
&= \int_X E(\cdot)T_1KJ\phi_-(y_2, \bar{w}_2)J_x(y_2)m(x, \hat{w}_2)dy_2\\
&= \int_X \int_X \int_{\Omega_{y_1}X} E(\cdot)k(y_2, \hat{w}_1, \bar{w}_2)k(y_1, \hat{w}, \bar{w}_1)\phi_-\big(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1, \hat{w}))\big)d\hat{w}\\
&\qquad \times J_{y_2}\, dy_1 J_x(y_2)m(x, \hat{w}_2)dy_2\\
&= \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \int_X E(\cdot)k(\vec{\gamma}_{(x',v')}(t), \bar{w}_1)k(y_2, \hat{w}_1, \bar{w}_2)\mathcal{J}(\gamma_{(x',v')}(t), y_2, x)\\
&\qquad \times m(x, \hat{w}_2)dy_2\, dt\, \phi_-(x', v')d\mu(x', v').
\end{aligned}$$

This proves (10) for $j = 2$.

An inductive argument on j first shows that

$$\begin{aligned}
T_1K^{j-1}J\phi_-(y_j, \bar{w}_j) &= \int_X \cdots \int_X \int_{\Omega_{y_1}X} E(\cdot)k(y_1, \hat{w}, \bar{w}_1)\prod_{i=2}^{j} k(y_i, \hat{w}_{i-1}, \bar{w}_i)\\
&\qquad \times \phi_-\big(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1, \hat{w}))\big)\mathcal{J}(y_1, \ldots, y_j)d\hat{w}\, dy_1 \ldots dy_{j-1}
\end{aligned}$$

where $\bar{w}_i$, $\hat{w}_i$ are the initial and final tangent vectors of the geodesic joining y_i to y_{i+1} for $i = 1, \ldots, j-1$, and where $\mathcal{J}$ is as in (6). Combining this with an inductive argument based on the above proof of (10) when $j = 2$ yields (10) for arbitrary $j \geq 2$. □

5. Recovering the Extinction Coefficient

We recall some definitions relevant to this section. The open set $\mathcal{H}_\sigma \subset \Gamma_-$ is such that the geodesic X-ray transform restricted to the set of geodesics with initial data in $\mathcal{H}_\sigma$ is injective. The notation indicates that this is the subset of geodesics which is used in the recovery of the extinction coefficient σ. The weight function m is assumed to be non-zero on Γ_+, and $m \in L^1(\mathcal{H}_\sigma)$. The main result of this section is Theorem 5.1 which shows that, due to the singular nature of α_0, we can use an approximate identity as our prescribed flux to obtain the extinction coefficient in a limiting process. The key estimate used in the proof of Theorem 5.1 is that of Proposition 5.1 which enables us to show that the contribution due to α_j for $j \geq 1$ tends to zero.

Let $0 < \mu < 1$ and set $p = (n - \mu)/(n - 1)$, $q = (1 - 1/p)^{-1}$.

Lemma 5.1. *It holds that* $d(x,\cdot)^{n-1} \in L^p(X)$ *with*

$$\left\| \frac{1}{d(x,\cdot)^{n-1}} \right\|_{L^p(X)} \leq C_d^{1/p}. \tag{11}$$

Proof. We compute

$$\int_X \frac{1}{d(x,y)^{(n-1)p}} dy = \int_X \frac{1}{d(x,y)^{n-\mu}}\, dy = \int_{\Omega_x X} \int_0^{\tau_+(x,v')} \frac{1}{t^{n-\mu}} J_x(\gamma_{(x,v')}(t))^{-1}\, dt\, dv'$$
$$\leq (C_{\kappa_m})^{n-1} \int_{\Omega_x X} \int_0^{\tau_+(x,v')} t^{\mu-1}\, dt\, dv' \leq C_d, \quad \text{say},$$

by (8). □

Definition 5.1. Let $\mathcal{T}$ be the operator with kernel $d(x,\cdot)^{n-1}$, that is

$$\mathcal{T} f(x) := \int_X \frac{f(y)}{d(x,y)^{n-1}}\, dy.$$

Lemma 5.1 (the proof is valid also when $p = 1$) shows that $\mathcal{T} : L^p(X) \to L^p(X)$ continuously, $1 \leq p \leq \infty$ with $\|\mathcal{T}\| \leq (C_{\kappa_m})^{n-1} |\mathbb{S}^{n-1}| \operatorname{diam} X$ (see [23], Prop. 5.1, Appendix A).

We also define the analogous operator $\widetilde{\mathcal{T}}$,

$$\widetilde{\mathcal{T}} f(x) := \int_X f(y) J_y(x) dy. \tag{12}$$

Since

$$\int_X J_y(x) dy = \int_{\Omega_x X} \int_0^{\tau_+(x,v)} dt\, dv \leq |\mathbb{S}^{n-1}| \operatorname{diam} X, \tag{13}$$

$\widetilde{\mathcal{T}} : L^p(X) \to L^p(X)$, $1 \leq p \leq \infty$, with $\|\widetilde{\mathcal{T}}\| \leq |\mathbb{S}^{n-1}| \operatorname{diam} X$.

Proposition 5.1. *Let* $p = (n-\mu)/(n-1)$, $0 < \mu < 1$, $q = (1 - 1/p)^{-1}$, *and* $\|k\|_{L^\infty(\Omega^2 X)} < \left[(C_{\kappa_m} C_{\kappa_M})^{n-1} \operatorname{diam} X |\mathbb{S}^{n-1}|\right]^{-1}$, *where* C_{κ_m}, C_{κ_M} *are as defined in Lemma* 4.2. *Then for almost every* $x \in \partial X$,

$$\left| \sum_{j=1}^{\infty} \int_{\Gamma_-} \alpha_j(x, x', v') f(x', v') d\mu(x', v') \right| \leq C_0 \|f\|_{L^q(\Gamma_-, d\mu)} \tag{14}$$

where $C_0 > 0$ *depends on* κ_m, κ_M, $\|k\|_{L^\infty(\Omega^2 X)}$, $\|m\|_{L^\infty(\Omega X)}$ *and* diam X.

Proof. When $j = 1$,

$$\left| \int_{\Gamma_-} \alpha_1(x, x', v') f(x', v') d\mu(x', v') \right|$$
$$= \left| \int_{\Gamma_-} \int_0^{\tau_+(x',v')} k(\vec{\gamma}_{(x',v')}(t), \bar{w}) E(\cdot) m(x, \hat{w}_1) J_x(\gamma_{(x',v')}(t)) dt\, f(x', v') d\mu(x', v') \right|$$

$$\le \|k\|_\infty \|m\|_\infty (C_{\kappa_M})^{n-1} \int_X \int_{\Omega_y X} \frac{|f(\vec{\gamma}_{(y,v)}(-\tau_-(y,v)))|}{d(x,y)^{n-1}}\, dv\, dy \qquad \text{by (7)}$$

$$\le (C_{\kappa_M})^{n-1} \|k\|_\infty \|m\|_\infty \left(\int_{\Omega X} \frac{1}{d(x,y)^{(n-1)p}}\, dv\, dy \right)^{\frac{1}{p}}$$

$$\times \left(\int_{\Omega X} \left| f(\vec{\gamma}_{(y,v)}(-\tau_-(y,v))) \right|^q dv\, dy \right)^{\frac{1}{q}} \qquad \text{by Hölder's inequality}$$

$$\le (C_{\kappa_M})^{n-1} \|k\|_\infty \|m\|_\infty |\mathbb{S}^{n-1}|^{1/p} C_d^{1/p} (\operatorname{diam} X)^{1/q} \|f\|_{L^q(\Gamma_-, d\mu)} \qquad \text{by (11)}$$

$$= C \|k\|_\infty \|m\|_\infty (\operatorname{diam} X)^{1/q} \|f\|_{L^q(\Gamma_-, d\mu)}, \quad \text{say.}$$

The terms for $j \ge 2$ are slightly different from the above; first when $j = 2$,

$$\left| \int_{\Gamma_-} \alpha_2(x, x', v') f(x', v') d\mu(x', v') \right|$$

$$= \left| \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \int_X E(\cdot)^2 k(\cdot)^2 m(\cdot) J_{y_2}(\gamma_{(x',v')}(t)) J_x(y_2) dt\, f(x', v') d\mu(x', v') \right|$$

$$\le (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 \int_X \int_{\Omega_{y_1} X} \int_X \frac{\left| f(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))) \right|}{d(y_1,y_2)^{n-1} d(y_2,x)^{n-1}}\, dy_2\, dv_1\, dy_1$$

$$= (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 \int_X \int_{\Omega_{y_1} X} \left| f(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))) \right| (\mathcal{T} h)(y_1) dv_1\, dy_1$$

where $h(y_2) = d(y_2, x)^{1-n}$,

$$\le (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 \left(\int_{\Omega X} |(\mathcal{T} h)(y_1)|^p\, dv_1\, dy_1 \right)^{\frac{1}{p}}$$

$$\times \left(\int_{\Omega X} \left| f(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))) \right|^q dv_1\, dy_1 \right)^{\frac{1}{q}} \qquad \text{by Hölder's inequality,}$$

$$\le (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 |\mathbb{S}^{n-1}|^{1/p} \|\mathcal{T} h\|_{L^p(X)} (\operatorname{diam} X)^{1/q} \|f\|_{L^q(\Gamma_-, d\mu)}$$

$$\le (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 |\mathbb{S}^{n-1}|^{1/p} \|(C_{\kappa_m})^{n-1} |\mathbb{S}^{n-1}| (\operatorname{diam} X) C_d^{1/p}$$

$$\times (\operatorname{diam} X)^{1/q} \|f\|_{L^q(\Gamma_-, d\mu)}$$

by (11). Finally, for any $j \ge 2$,

$$\left| \int_{\Gamma_-} \alpha_j(x, x', v') f(x', v') d\mu(x', v') \right|$$

$$= \left| \int_{\Gamma_-} \int_0^{\tau_+(x',v')} \int_X \cdots \int_X E(\cdot)^{j+1} k(\cdot)^j \mathcal{J}(\gamma_{(x',v')}(t), y_2, \ldots, y_j, x) \right.$$

$$\left. \times\, m(\cdot) dy_j \ldots dy_2\, dt\, f(x', v') d\mu(x', v') \right|$$

$$\le (C_{\kappa_M})^{j(n-1)} \|m\|_\infty \|k\|_\infty^j$$

$$\times \int_X \int_{\Omega_{y_1}} \int_X \cdots \int_X \frac{\left|f\left(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\right)\right|}{d(y_j,x)^{n-1}\prod_{l=1}^{j-1} d(y_l,y_{l+1})^{n-1}} dy_j \dots dy_2\, dv_1\, dy_1$$

$$= (C_{\kappa_M})^{j(n-1)} \|m\|_\infty \|k\|_\infty^j \int_X \int_{\Omega_{y_1}} \left|f\left(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\right)\right| (T^{j-1}h)(y_1) dv_1\, dy_1$$

where $h(y) = d(y,x)^{1-n}$,

$$\leq (C_{\kappa_M})^{j(n-1)} \|m\|_\infty \|k\|_\infty^j |\mathbb{S}^{n-1}|^{1/p} \|T^{j-1}h\|_{L^p(X)}$$
$$\times \left(\int_{\Omega X} \left|f\left(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\right)\right|^q dv_1\, dy_1 \right)^{\frac{1}{q}}$$
$$\leq (C_{\kappa_M})^{j(n-1)} \|m\|_\infty \|k\|_\infty^j \left[(C_{\kappa_m})^{n-1} (\operatorname{diam} X) |\mathbb{S}^{n-1}|\right]^{j-1} C_d^{1/p}$$
$$\times (\operatorname{diam} X)^{1/q} \|f\|_{L^q(\Gamma_-, d\mu)}$$
$$= C\left[\|k\|_\infty (C_{\kappa_M} C_{\kappa_m})^{n-1} (\operatorname{diam} X) |\mathbb{S}^{n-1}|\right]^{j-1} \|f\|_{L^q(\Gamma_-, d\mu)}$$

where $C = \left[C_d^{1/p} (\operatorname{diam} X)^{1/q} (C_{\kappa_M})^{n-1} \|m\|_\infty \|k\|_\infty\right]$. Our smallness assumption on k ensures that the sum converges, and we have the claim of the proposition. □

Before proceeding to Theorem 5.1 we must construct an appropriate approximate identity to use as our in-going boundary flux. Let $x \in X$ be fixed; $\Omega_x X$ is the unit sphere in $T_x X$ with respect to the metric $g(x)$ on $T_x X$. Endow $\Omega_x X$ with the metric induced from the inner product on $T_x X$. Fix $v \in \Omega_x X$ and let $\exp_v : T_v \Omega_x X \to \Omega_x X$ be the exponential map for $\Omega_x X$ based at v. If $\hat{w} = \hat{w}(w) = \exp_v^{-1}(w)$ let $\mathcal{J}_{(x,v)}(\hat{w})$ be the determinant of the Jacobian of this change of variables.

Let $\varphi \in C_c^\infty(\mathbb{R}^{n-1})$ be such that $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\varphi(\hat{w}) = 0$ for $|\hat{w}| \geq \varepsilon_0$ (ε_0 fixed and sufficiently small), and $\int_{\mathbb{R}^{n-1}} \varphi(\hat{w}) d\hat{w} = 1$. Define $\psi_\eta : \Omega_x X \to \mathbb{R}$ by

$$\psi_\eta(w) = \frac{1}{\eta^{n-1}} \varphi\left(\frac{\exp_v^{-1}(w)}{\eta}\right).$$

Note that if $f : \Omega_x X \to \mathbb{R}$ is continuous at v then

$$\int_{\Omega_x X} f(w)\psi_\eta(w) dw = \int_{\mathbb{R}^{n-1}} f(\exp_v(\hat{w})) \frac{1}{\eta^{n-1}} \varphi\left(\frac{\hat{w}}{\eta}\right) \mathcal{J}_{(x,v)}(\hat{w}) d\hat{w}$$
$$\to f(\exp_v(0)) \mathcal{J}_{(x,v)}(0) = f(v)$$

as $\eta \to 0$. Now we will use such functions to construct $f_\eta \in L^1(\Gamma_-)$. Fix a point $(x^*, v^*) \in \Gamma_+$. Define $\psi_{x^*,\eta} : \Omega^+_{x^*} X \to \mathbb{R}$ by

$$\psi_{x^*,\eta}(v) = \frac{1}{\eta^{n-1}} \varphi\left(\frac{\exp_{v^*}^{-1}(v)}{\eta}\right).$$

Let $\rho \in C_c^\infty(\mathbb{R})$ be such that $0 \leq \rho \leq 1$, $\rho(0) = 1$, and $\rho(t) = 0$ for $|t| \geq \varepsilon_0$; for $\lambda > q - 1$ define $\chi_\eta : \partial X \to \mathbb{R}$ by

$$\chi_\eta(x) = \rho\left(\frac{d_{\partial X}(x, x^*)}{\eta^\lambda}\right)$$

where $d_{\partial X}(x, y)$ is distance in the boundary of X. Now if we denote by $\mathscr{P}(v; x, x^*)$ the parallel translation of v along the geodesic (in X) joining x to x^*, then we can define $\psi_{x,\eta} : \Omega_x X \to \mathbb{R}$ by

$$\psi_{x,\eta}(v) = \psi_{x^*,\eta}(\mathscr{P}(v; x, x^*)).$$

This is simply a smooth way of extending the definition of $\psi_{x^*,\eta}$ from $\Omega^+_{x^*}X$ to $\Omega_x X$ for x near x^*. Finally, define $f_\eta \in L^1(\Gamma_-)$ by

$$f_\eta(x', v') = \chi_\eta\big(\gamma_{(x',v')}(\tau_+(x', v'))\big)\psi_{\gamma_{(x',v')}(\tau_+(x',v')),\eta}\big(\dot{\gamma}_{(x',v')}(\tau_+(x', v'))\big). \tag{15}$$

Note that if $(x^*, v^*) \in \mathscr{H}_\sigma$ then for sufficiently small η, f_η is supported in $\mathscr{H}_\sigma$. In the following theorem we show that $\mathscr{M}$ determines the integral of σ along almost every given geodesic of (X, g) with data in $\mathscr{H}_\sigma$, and hence the geodesic X-ray transform of σ, restricted to these geodesics. By definition of $\mathscr{H}_\sigma$, this uniquely determines the function σ, and proves Theorem 2.1.

Theorem 5.1. *Let* (σ, k, m) *satisfy the hypothesis of Theorem* 2.1. *Let* $f_\eta \in L^1(\Gamma_-)$ *be as defined in* (15) *above. Then for almost every* $(x^*, v^*) \in \mathscr{H}_\sigma$

$$\lim_{\eta\to 0} \mathscr{M}f_\eta(x^*) = E\big(\gamma_{(x^*,v^*)}(-\tau_-(x^*, v^*)), x^*\big)m(x^*, v^*).$$

Proof. Let $v^* \in \Omega^+_{x^*}X$ be such that Theorem 4.1 holds at x^* (this happens *a.e.*) and such that (x^*, v^*) is in the Lebesgue set of $E\big(\gamma_{(x^*,v^*)}(-\tau_-(x^*, v^*)), x^*\big)m(x^*, v^*)$. From Theorem 4.1 we have $\mathscr{M}f_\eta = \sum_{j=0}^\infty \int_{\Gamma_-} \alpha_j f_\eta \, d\mu$. When $j = 0$,

$$\begin{aligned}
&\int_{\Gamma_-} \alpha_0(x^*, x', v')f_\eta(x', v')d\mu \\
&= \int_{\Omega^+_x X} E\big(x^*, \gamma_{(x^*,v)}(-\tau_-(x^*, v))\big)f_\eta\big(\vec{\gamma}_{(x^*,v)}(-\tau_-(x^*, v))\big)m(x^*, v)dv \\
&= \int_{\Omega^+_{x^*} X} E\big(x^*, \gamma_{(x^*,v)}(-\tau_-(x^*, v))\big)\chi_\eta(x^*)\psi_{x^*,\eta}(v)m(x^*, v)dv \\
&\to E\big(\gamma_{(x^*,v^*)}(-\tau_-(x^*, v^*)), x^*\big)m(x^*, v^*)
\end{aligned}$$

as $\eta \to 0$ since (x^*, v^*) is in the Lebesgue set of the limiting function. It remains to show that the rest of the series tends to zero. Changing variables $(x, v) = \vec{\gamma}_{(x',v')}(\tau_+(x', v'))$ (it is easy to check that the volume elements are the same), we compute

$$\begin{aligned}
\|f_\eta\|^q_{L^q(\Gamma_-,d\mu)} &= \int_{\Gamma_-} |f_\eta(x', v')|^q \, d\mu = \int_{\Gamma_+} |f_\eta(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))|^q \, d\mu \\
&= \int_{\partial X}\int_{\Omega^+_x X} \chi_\eta(x)^q \psi_{x,\eta}(v)^q |\langle v, \nu_x\rangle| dv \, dx \\
&\le C\int_{\partial X}\int_{\Omega^+_x X} \chi_\eta(x)^q \psi_{x,\eta}(v)^q \, dv \, dx \\
&= C\int_{\partial X} \chi_\eta(x)^q \int_{\Omega^+_x X} \frac{1}{\eta^{(n-1)q}}\varphi\left(\frac{\exp^{-1}_{v^*}(\mathscr{P}(v; x, x^*))}{\eta}\right)^q \, dv \, dx.
\end{aligned}$$

In a sufficiently small neighborhood $N \subset \partial X$ of x^*, we shall make the change of variables $(x, v) \mapsto (x, \hat{w}_x)$ where $(x, v) \in \bigcup_{x\in N} \Omega_x X$, defined by

$$(x, \hat{w}_x) = \big(x, \exp_{v^*}^{-1}(\mathscr{P}(v; x, x^*))\big) \in \partial X \times \mathbb{R}^{n-1}$$

where $\exp_{v^*} : T_{v^*}\Omega_{x^*}X \to \Omega_{x^*}X$. This is a smooth map and so the Jacobian determinant is bounded on N by M, say. We thus have

$$\begin{aligned}
\|f_\eta\|^q_{L^q(\Gamma_-,d\mu)} &\leq CM \int_{\partial X} \chi_\eta(x)^q \int_{\mathbb{R}^{n-1}} \frac{1}{\eta^{(n-1)q}} \varphi\left(\frac{\hat{w}_x}{\eta}\right)^q d\hat{w}_x\, dx \\
&= CM \int_{\partial X} \chi_\eta(x)^q \int_{\mathbb{R}^{n-1}} \frac{1}{\eta^{(n-1)(q-1)}} \varphi(\hat{w})^q\, d\hat{w}\, dx \\
&\leq M' \frac{1}{\eta^{(n-1)(q-1)}} \int_{\partial X} \rho\left(\frac{d_{\partial X}(x, x*)}{\eta^\lambda}\right)^q dx \\
&\leq M' \frac{1}{\eta^{(n-1)(q-1)}} \mathrm{Vol}_{\partial X}(\{x \in \partial X : d_{\partial X}(x, x*) < \varepsilon_0 \eta^\lambda\}) \\
&\leq M' \frac{1}{\eta^{(n-1)(q-1)}} \mathrm{Vol}_{\mathbb{R}^{n-1}}(\{|x| < \varepsilon_0 \eta^\lambda\}) \\
&= \frac{M' |\mathbb{S}^{n-2}| \varepsilon_0^{n-1}}{n-1} \eta^{\lambda(n-1)-(n-1)(q-1)} \to 0
\end{aligned}$$

as $\eta \to 0$ for any $\lambda > q - 1$.

To complete the proof of the theorem, by Proposition 5.1, we have

$$\left|\sum_{j=1}^{\infty} \int_{\Gamma_-} \alpha_j(x, x', v') f_\eta(x', v') d\mu(x', v')\right| \leq C_0 \|f_\eta\|_{L^q(\Gamma_-,d\mu)} \to 0$$

as $\eta \to 0$ from the above computation. □

6. Recovering the Scattering Kernel

Throughout this section, the measurement point x will be fixed. We assume here that the scattering kernel is less general than in the previous section. Precisely, we assume that it is of the form $k(x)\Theta(x, v', v)$, where $\Theta(x, v', v)$ is *a-priori* known. We prove in this setting that the spatial distribution $k(x)$ is uniquely determined by the averaged albedo operator $\mathscr{M}$, and in fact prove that this is so from knowledge of measurements at the single fixed measurement point x.

Definition 6.1. Given a complete Riemannian manifold (X, g) with geodesics $\gamma_{(x,v)}(t)$, and functions $\eta : \Omega X \to \mathbb{R}$, and $\beta \in C^\infty(\Gamma_-)$ we may define the *weighted geodesic transform* by, for $f : X \to \mathbb{R}$,

$$I_{\eta,\beta} f(x', v') := \beta(x', v') \int_0^{\tau_+(x',v')} f(\gamma_{(x',v')}(t))\eta(\vec{\gamma}_{(x',v')}(t))dt. \tag{16}$$

We also have the L^2 adjoint, for $f : \Gamma_- \to \mathbb{R}$,

$$I^*_{\eta,\beta} f(x) = \int_{\Omega_x X} f\big(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))\big)\beta\big(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))\big)\eta(x, v)dv. \tag{17}$$

Under appropriate assumptions to be stated shortly, the kernel α_1 which represents single scattering is a weighted X-ray transform of the unknown function $k(x)$. We make use of the injectivity results of [10] for such an X-ray transform to prove unique identifiability of k. The results of [10] require a sufficiently rich set of curves (geodesics in our case) along which the integral transform is known. The inclusion of the factor β in definition (16) essentially serves the purpose of restricting the transform to a (possibly proper) subset of geodesics. This is made clearer in the following definition:

Definition 6.2. We say that Γ is a regular family of curves (for the metric g) if for any $(x, v) \in T^*X\backslash\{0\}$ there exists $\gamma \in \Gamma$ through x, normal to v, and such that γ has no conjugate points.

We say that $\beta \in C^\infty(\Gamma_-)$ is regular if there exists a set $\mathscr{H} \subset \{(x', v')\beta(x', v') \neq 0\}$ such that $\Gamma(\mathscr{H})$ is a regular family.

Note that in our setting of simplicity of (X, g), all geodesics are without conjugate points. In dimension $n = 2$, one must take all geodesics of X in order to satisfy regularity as defined in Definition 6.2; when $n \geq 3$, one may take proper subsets of the set of all geodesics and still satisfy regularity.

The following two theorems are proven (in greater generality) in [10].

Theorem 6.1 ([10]). *Let $\beta \in C^\infty(\Gamma_-)$ be a regular function for the real-analytic metric g, as in Definition 6.2. Suppose that $\eta : \Omega X \to \mathbb{R}$ is real-analytic and non-vanishing on $\overline{U}$ where $U \subset \overline{U} \subset X$. Then $I_{\eta,\beta}$ is injective on $L^1(U)$.*

Theorem 6.2 ([10]). *Let β and U be as in Theorem 6.1, and let $\eta \in C^\infty(\Omega X)$. Then we may find a constant C such that:*

(a) *If $I_{\eta,\beta}$ is injective on $L^2(U)$, then*

$$\frac{1}{C}\|f\|_{L^2(U)} \leq \|I^*_{\eta,\beta} I_{\eta,\beta} f\|_{H^1(X)} \leq C\|f\|_{L^2(U)}.$$

(b) *There exists a C^2 neighborhood of (η, β), and a C^3 neighborhood of g on which the above estimate remains true, with a uniform constant C.*

To apply these results to our problem, we first recall that $k(x)\Theta(x, v, v')$ and $\tilde{k}(x)\Theta(x, v, v')$ are two scattering kernels with $k, \tilde{k} \in \mathscr{K}^D_\varepsilon$; i.e., $\operatorname{dist}(\operatorname{supp}(k), \partial X) > D$ and $\|k\|_{L^\infty(X)} \leq \varepsilon$, and similarly for $\tilde{k}$. Let $\mathscr{M}$, $\tilde{\mathscr{M}}$, and α, $\tilde{\alpha}$ be the averaged albedo operators and Schwarz kernels associated to k and $\tilde{k}$ respectively. We set $\Delta k = k - \tilde{k}$ and $\Delta\alpha_j = \alpha_j - \tilde{\alpha}_j$, $j = 1, 2, \ldots$.

For an appropriately defined weight w, we have $\alpha_1(x, x', v') = I_{w,1}k(x', v')$. To this end, let $\chi \in C_c^\infty(X)$ with $\chi \equiv 1$ on a neighborhood of $\{y \in X : \operatorname{dist}(y, \partial X) \geq D\}$. If $x_1 \in X$, let $\bar{v} = \bar{v}(x_1) \in \Omega_{x_1}X$ and $v = v(x_1) \in \Omega_x^+ X$ be the initial and final tangent vectors, respectively, of the geodesic joining x_1 to x. Then for $v_1 \in \Omega_{x_1}X$ define the weight function

$$w(x_1, v_1) := \Theta(x_1, v_1, \bar{v})E\big(\gamma_{(x_1,v_1)}(-\tau_-(x_1, v_1)), x_1, x\big)m(x, v)J_x(x_1)\chi(x_1). \tag{18}$$

With this definition, we see from (9) and (16) that indeed $\alpha_1(x, x', v') = I_{w,1}k(x', v')$. Assuming that the metric g is real-analytic, J_x is then real-analytic; if further

Θ, σ and m are real-analytic, then w is a real-analytic, non-vanishing weight function in $\{y \in X : \mathrm{dist}(y, \partial X) \geq D\}$. If $\beta \in C^\infty(\Gamma_-)$ is a regular function for g then Theorems 6.1 and 6.2 give us:

Lemma 6.1. *Let (m, Θ, σ, g) be fixed and real analytic, and suppose $\beta \in C^\infty(\Gamma_-)$ is a regular function for g. Then there exits $C > 0$, independent of k, $\tilde{k} \in \mathscr{K}_\varepsilon^D$ such that*

$$\|\Delta k\|_{L^2(X)} \leq C\|I^*_{w,\beta} I_{w,\beta} \Delta k\|_{H^1(X)}, \tag{19}$$

with the above estimate holding in a C^2 neighborhood of (m, Θ, σ), and a C^3 neighborhood of g.

Proposition 6.1. *Let (m, Θ, σ, g) be fixed and real analytic, and suppose $\beta \in C^\infty(\Gamma_-)$ is a regular function for g. Furthermore, suppose $\mathscr{H}_k \subset \{(x', v') : \beta(x', v') \neq 0\}$ is such that $\Gamma(\mathscr{H}_k)$ is a regular set of geodesics. Suppose that $\mathscr{M} = \widetilde{\mathscr{M}}$ on $L^1(\mathscr{H}_k, d\mu)$. Then there exists $C > 0$ such that for all k, $\tilde{k} \in \mathscr{K}_\varepsilon^D$*

$$\|\Delta k\|_{L^2(X)} \leq C\|I^*_{w,\beta} I_{w,\beta} \Delta k\|_{H^1(X)} = C\|I^*_{w,\beta} \beta \Delta\alpha_1\|_{H^1(X)} = C\left\| I^*_{w,\beta} \beta \sum_{j=2}^{\infty} \Delta\alpha_j \right\|_{H^1(X)},$$

with the above estimate holding in a C^2 neighborhood of (m, σ, Θ) and a C^3 neighborhood of g.

Proof. The first inequality is (19); next, $\beta I_{w,1} = I_{w,\beta}$ and w has been defined so that $I_{w,1}\Delta k = \Delta\alpha_1$; finally, since $\sigma = \tilde{\sigma}$, $\alpha_0 = \tilde{\alpha}_0$ and so $\mathscr{M} = \widetilde{\mathscr{M}}$ implies that $\sum_{j=1}^{\infty} \Delta\alpha_j = 0$ which proves the final equality. □

In order to prove unique identifiability of k we proceed to prove that $\|\Delta k\|_{L^2(X)} \leq \varepsilon C \|\Delta k\|_{L^2(X)}$, and so for sufficiently small $\varepsilon > 0$, we will have $\Delta k = 0$. We begin with Proposition 6.2 which is very similar to Proposition 4 in [10]. However, to use that result here we would have to assume that $(\eta_1, \beta_1, \eta_2, \beta_2)$ were all small.

Proposition 6.2. *There is $C > 0$ such that for all $f \in L^2(X)$ with $\mathrm{supp}\, f \subset \{y \in X : d(x, \partial X) > D\}$,*

$$\|I^*_{\eta_1,\beta_1} I_{\eta_2,\beta_2} f\|_{H^1(X)} \leq C\|\eta_1\|_{C^2(\Omega X)} \|\eta_2\|_{C^2(\Omega X)} \|\beta_1\beta_2\|_{C^2(\Gamma_-)} \|f\|_{L^2(X)}, \tag{20}$$

with C depending continuously on the C^4 norm of g.

Proof. The proof is very similar to that of Proposition 4 in [10] and we refer the reader there for some additional details. We will compute in a fixed global coordinate system $\hat{x} = \{\hat{x}^1, \ldots, \hat{x}^n\}$ for X. Let $\hat{v} = \{\hat{v}_1, \ldots, \hat{v}_n\} = \{\partial_{\hat{x}^1}, \ldots, \partial_{\hat{x}^n}\}$ be the naturally induced coordinates on T_xX. We define C^s and H^s norms with respect to this fixed coordinate system.

We shall also need to change to coordinates in which for a fixed $x \in X$, $\Omega_x X$ is $\mathbb{S}^{n-1}$. To this end, let $x \in X$ and given $v \in \Omega_x X$ and $0 \leq t \leq \tau_+(x, v)$ let $\hat{y} = \hat{y}(t, v; x) \in \mathbb{R}^n$ be the coordinate expression for $y = \exp_x(tv)$. Define $\omega \in \mathbb{S}^{n-1}$

by $\omega = \omega(t, v; x) = (\hat{y} - \hat{x})/|\hat{y} - \hat{x}|$ and $r = r(t, v; x) \geq 0$ by $r = |\hat{y} - \hat{x}|$. If $\hat{v}(v)$ is the coordinate expression of v, we define $\omega(0, v) = \hat{v}/|\hat{v}|$. We claim that the map $(t, v) \mapsto (r, \omega)$ is smooth, including at $t = 0$, and that for sufficiently small $t > 0$ it is a diffeomorphism onto its range. It is convenient (as in [10]) to define $m(t, v; x) := (\hat{y}(t, v) - \hat{x})/t$. Then it holds that $m(0, v; x) := \lim_{t\to 0} m(t, v; x) = \hat{v} \neq 0$ and expanding $\hat{y}$ in a Taylor series about $t = 0$ one obtains $\omega = m/|m|$ and $r = t|m|$ are smooth, including at $t = 0$.

We now demonstrate that at $t = 0$ (and hence in a neighborhood) the map is of full rank. In coordinates, the map $\{\sum_{ij} \hat{v}_i g_{ij}(x)\hat{v}_j = 1\} \ni \hat{v} \mapsto \omega \in \mathbb{S}^{n-1}$ is given by $\omega := G^{1/2}\hat{v}$ where for every x, $G^{1/2}(x)$ is a square root of the matrix given by the metric. We denote by $\frac{\partial\omega}{\partial v}$ the differential of this map, and use the same notation for the $(n-1)\times(n-1)$ matrix representation in a choice of coordinates. One has $\partial_t r|_{t=0} = |\hat{v}|$ and $\partial_v r|_{t=0} = 0$ so that in the $(\hat{x}, \hat{v})$ coordinate system,

$$\frac{\partial(\omega, r)}{\partial(v, t)}\bigg|_{t=0} = \begin{pmatrix} \frac{\partial\omega}{\partial v} & \frac{\partial\omega}{\partial t} \\ 0 & |\hat{v}| \end{pmatrix}$$

is full-rank. Thus, $|\det\frac{\partial(\omega,r)}{\partial(v,t)}| \neq 0$ at $t = 0$ and there exists $\varepsilon_1(x)$ such that for $0 \leq t < \varepsilon_1(x)$ the change of variables $\psi : (t, v) \mapsto (r, \omega)$ is a diffeomorphism onto its range. However, the domain in $\mathbb{R}^n$ described by the polar coordinates $(r, w) \in \psi([0, \varepsilon_1(x)) \times \mathbb{S}^{n-1})$ need not be star-shaped with respect to the origin; or put another way, if $(r_0, \omega_0) \in \psi([0, \varepsilon_1(x)) \times \mathbb{S}^{n-1})$, it is not necessarily true that the same holds for all $0 \leq r \leq r_0$. But there does exist $r_m(x) > 0$ such that for all $\omega \in \mathbb{S}^{n-1}$ and $0 \leq r \leq r_m(x)$ we have $(r, \omega) \in \psi([0, \varepsilon_1(x)) \times \mathbb{S}^{n-1})$. Let $r_m(x)$ be the largest such radius for which this holds. Then it is clear that there exists $0 < R_m \leq r_m(x)$ for all x; indeed, let $\{B_x\}$ be an open cover of the compact set $\{x \in X : \text{dist}(x, \partial X) \geq D/2\}$ by balls of radius $r_m(x)$, and let R_m be the Lebesgue number associated with that cover. We then have that any ball of radius $\leq R_m$ is contained in one member of this cover, and therefore in a convex neighborhood which is the image of the diffeomorphism ψ.

With these preparations complete, let f be supported as in the statement of the proposition. Then, re-parameterizing in the t variable,

$$I^*_{\beta_1,\eta_1} I_{\beta_2,\eta_2} f(x) = \int_{\Omega_x X} \int_{-\tau_-(x,v)}^{\tau_+(x,v)} A(x, t, v) f(\gamma_{(x,v)}(t))\,dt\,dv = (I_1), \quad \text{say},$$

with $A(x, t, v) := \eta_1(x, v)\eta_2(\vec{\gamma}_{(x,v)}(t))\beta_1\big(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))\big)\beta_2\big(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))\big)$. If $A(x, t, \cdot)$ is odd, then it is easily seen that the integral vanishes and so if $A_e(x, t, v) := A(x, t, v) + A(x, t, -v)$,

$$(I_1) = \int_{\Omega_x X} \int_0^{\tau_+(x,v)} \chi(t) A_e(x, t, v) f(\gamma_{(x,v)}(t))\,dt\,dv$$
$$+ \int_{\Omega_x X} \int_0^{\tau_+(x,v)} (1 - \chi(t)) A_e(x, t, v) f(\gamma_{(x,v)}(t))\,dt\,dv = (I_2) + (I_3), \quad \text{say},$$

where $\chi(t) \in C^\infty(\mathbb{R})$, $\chi(0) = 1$, and is supported in $[0, \delta)$ with $\delta < \min\{R_m, D\}$. Rewriting (I_3) in terms of spatial variables we easily obtain

$$\|(I_3)\|_{H^1(X)} \leq C \sup_{x,y\in X,\ d(x,y)\geq\delta} \sum_{|\alpha|\leq 1} \left|\partial_{\hat{x}}^\alpha A_e(\hat{x}, d(\hat{x},\hat{y}), \hat{v}(\hat{x},\hat{y}))J_{\hat{x}}(\hat{y})\right| \|f\|_{L^2(X)}, \tag{21}$$

with C depending only on $\mathrm{Vol}(X)$, which depends continuously on the metric. Here $v(x, y)$ is the initial tangent vector of the geodesic joining x to y, and $J_x(y)$ is the Jacobian determinant of the change of variables. Note that the Jacobian J_x depends continuously on the magnitude of Jacobi fields, which in turn depend continuously on one derivative of the Christoffel symbols, which in turn depend on one derivative of the metric. Hence

$$\begin{aligned}\|(I_3)\|_{H^1(X)} &\leq C\|A_e\|_{C^1}\|f\|_{L^2(X)} \\ &\leq C\|\eta_1\|_{C^1(\Omega X)}\|\eta_2\|_{C^1(\Omega X)}\|\beta_1\beta_2\|_{C^1(\Gamma_-)}\|f\|_{L^2(X)}.\end{aligned}$$

To treat (I_2), we express the integral in terms of our chosen coordinates. Let $\hat{y}$ denote the coordinate form of $\gamma_{(x,v)}(t)$, $d\hat{v}$ be the volume form dv expressed in these coordinates, and $\hat{\mathbb{S}} = \{\hat{v} : \hat{v}^i g_{ij}(x)\hat{v}^j = 1\} \subset \mathbb{R}^n$. Then changing coordinates to transform $\hat{\mathbb{S}}$ to $\mathbb{S}^{n-1}$ and letting $d\omega$ be the standard volume form on $\mathbb{S}^{n-1}$,

$$(I_2) = \int_{\mathbb{S}^{n-1}} \int_0^\infty G_e(\hat{x}, r, \omega) f(\hat{x} + r\omega)\, dr\, d\omega$$

where

$$\begin{aligned}G_e &= \frac{1}{2}\big(G(\hat{x}, r, \omega) + G(\hat{x}, r, -\omega)\big), \\ G(\hat{x}, r, \omega) &= \chi(t)A_e(\hat{x}, t, \hat{v})J(\hat{x}, t, \hat{v})\big|_{t=t(\hat{x},r,\omega),\hat{v}=\hat{v}(\hat{x},r,\omega)}\end{aligned}$$

where J is the Jacobian determinant for the changes of variables from $\hat{v}$ to ω. Expanding in a Taylor series, $G_e(\hat{x}, r, \omega) = G_0(\hat{x}, \omega) + rG_1(\hat{x}, r, \omega)$. We are left needing to estimate $(I_2) = (I_4) + (I_5)$ where

$$(I_4) = \int_{\mathbb{S}^{n-1}} \int_0^\infty G_0(x, \omega) f(\hat{x} + r\omega)\, dr\, d\omega = \int_{\mathbb{R}^n} G_0\left(\hat{x}, \frac{\hat{y} - \hat{x}}{|\hat{y} - \hat{x}|}\right) \frac{f(\hat{y})}{|\hat{y} - \hat{x}|^{n-1}}\, d\hat{y},$$

and

$$\begin{aligned}(I_5) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty G_1(x, r, \omega) f(\hat{x} + r\omega)\, dr\, d\omega \\ &= \int_{\mathbb{R}^n} G_1\left(\hat{x}, |\hat{y} - \hat{x}|, \frac{\hat{y} - \hat{x}}{|\hat{y} - \hat{x}|}\right) \frac{f(\hat{y})}{|\hat{y} - \hat{x}|^{n-2}}\, d\hat{y}.\end{aligned}$$

We consider (I_5) first: we may differentiate inside the integral in (I_5) obtaining

$$\frac{\partial}{\partial \hat{x}^i}(I_5) = \int_{\mathbb{R}^n} \left(\left(\frac{\partial}{\partial \hat{x}^i} G_1(\cdot)\right) \frac{f(\hat{y})}{|\hat{y} - \hat{x}|^{n-2}} + G_1(\cdot) \frac{f(\hat{y})(\hat{y}^i - \hat{x}^i)}{|\hat{y} - \hat{x}|^n}\right) d\hat{y} \tag{22}$$

from which we have (in a manner similar to our bounds on $\mathcal{T}$, see definition 5.1)

$$\|(I_5)\|_{H^1(X)} \leq C\|G_1\|_{C^1(\Omega X)}\|f\|_{L^2(X)}, \tag{23}$$

the constant C depending only on Vol(X). Derivatives of the kernel of (I_4) yield a non-integrable kernel; however, the resulting integral makes sense as a principal value integral, and the mapping properties of such a singular integral operator are understood and studied in [16]. Indeed, differentiating (I_4) with respect to $\hat{x}^i$, treating ω and r as independent of $\hat{x}^i$, we obtain integrals of the same form as (22) and hence estimates analogous to (23) with G_1 replaced by G_0. Next, differentiating (I_4) assuming that *only* ω and r depend on $\hat{x}^i$ results in a kernel whose mean value in the ω variable is zero and whose $L^2(d\omega)$ norm is bounded by the $C^1(\Omega X)$ norm of G_0 (see [16] Section 7, Chapter IX). The Calderon–Zygmund theorem ([16], Theorem 3.1, Chapter XI) then yields

$$\|(I_4)\|_{H^1(X)} \leq C\|G_0\|_{C^1(\Omega X)}\|f\|_{L^2(X)}, \tag{24}$$

with C independent of G_0 and the metric. Combining the estimates (21), (23) and (24) above, $(I_1) = (I_4) + (I_5) + (I_3)$ satisfies

$$\begin{aligned}\|(I_1)\|_{H^1(X)} &\leq \|(I_3)\|_{H^1(X)} + C'(\|G_0\|_{C^1} + \|G_1\|_{C^1})\|f\|_{L^2(X)} \\ &\leq C''\|\eta_1\|_{C^2(\Omega X)}\|\eta_2\|_{C^2(\Omega X)}\|\beta_1\beta_2\|_{C^2(\Gamma_-)}\|f\|_{L^2(X)}.\end{aligned}$$

In a manner similar to the discussion after (21), C'' depends continuously on the C^4 norm of the metric. □

For each $i > 1$ we will write $\Delta\alpha_i$ as an infinite sum of weighted X-ray transforms; in order to have properly defined weights mapping $\Omega X \to \mathbb{R}$ we expand one instance of the kernel Θ occurring in $\Delta\alpha_i$ in a manner based on spherical harmonic expansions of functions on $\mathbb{S}^{n-1}$. We present the statement of the lemma demonstrating this expansion and postpone its proof until after we apply it to prove Proposition 6.3.

Lemma 6.2. *Let* $\Theta(x, v', v) \in C^{3n}(\Omega^2 X)$ *and* $g \in C^{3n}(X)$. *There exist* $\Theta_j(x, v')$, $\varphi_j(x, v) \in C^{3n}(\Omega X)$ *such that*

$$\Theta(x, v', v) = \sum_{j=1}^{\infty} \Theta_j(x, v')\varphi_j(x, v)$$

with

$$\|\Theta_j\|_{C^2(\Omega X)} \leq \frac{C}{1 + j^2}, \quad \textit{and} \quad \|\varphi_j(x, v)\|_{L^\infty(\Omega X)} \leq 1,$$

the above estimate holding in a C^{3n} *neighborhood of* (Θ, g).

Proposition 6.3. *Fix* (m, g, Θ, σ) *with* $m, \sigma \in C^2$, *and* $g, \Theta \in C^{3n}$. *Suppose that* $\|k\|_\infty, \|\tilde{k}\|_\infty < [\|\Theta\|_\infty \mathrm{diam}\, X|\mathbb{S}^{n-1}|]^{-1}$, $k, \tilde{k} \in \mathscr{K}_\varepsilon^D$, *and* $\beta \in C^\infty(\Gamma_-)$. *Then there is* $C > 0$ *such that*

$$\left\| I^*_{w,\beta} \beta \sum_{i=2}^{\infty} \Delta\alpha_i \right\|_{H^1(X)} \leq C\varepsilon \|\Delta k\|_{L^2(X)},$$

with the above estimate holding in a C^2 *neighborhood of* (m, σ), *and a* C^{3n} *neighborhood of* (Θ, g).

Proof. For $(x, v) \in \Omega X$ we define $\mathscr{E}(x, v) = E(\gamma_{(x,v)}(-\tau_-(x, v)), x)$. From Theorem 4.1,

$$\begin{aligned}\Delta\alpha_2(x, x', v') = \int_0^{\tau_+(x',v')} & E(x', \gamma_{(x',v')}(t)) \int_X \big[\Delta k(\gamma_{(x',v')}(t))k(y_2) - \tilde{k}(\gamma_{(x',v')}(t))\Delta k(y_2)\big] \\ & \times \Theta(\vec{\gamma}_{(x',v')}(t), \bar{w}_1)\Theta(y_2, \hat{w}_1, \bar{w}_2)E(\gamma_{(x',v')}(t), y_2) \\ & \times E(y_2, x)\mathscr{J}(\gamma_{(x',v')}(t), y_2, x)dy_2\, dt\end{aligned}$$

and from Lemma 6.2 we have (formally at least),

$$\begin{aligned}\Delta\alpha_2(x, x', v') &= \sum_{l=1}^{\infty} \int_0^{\tau_+(x',v')} \mathscr{E}(\vec{\gamma}_{(x',v')}(t))\Theta_l(\vec{\gamma}_{(x',v')}(t)) \\ &\quad \times \int_X \big[\Delta k(\gamma_{(x',v')}(t))k(y_2) + \tilde{k}(\gamma_{(x',v')}(t))\Delta k(y_2)\big] \\ &\quad \times \phi_l(\gamma_{(x',v')}(t), \bar{w}_1)\Theta(y_2, \hat{w}_1, \bar{w}_2) \\ &\quad \times E(\gamma_{(x',v')}(t), y_2)E(y_2, x)\mathscr{J}(\gamma_{(x',v')}(t), y_2, x)m(x, \hat{w}_2)dy_2\, dt \\ &= \sum_{l=1}^{\infty} I_{\Theta_l \mathscr{E},1}\big[\Psi_{2,1,l} + \Psi_{2,2,l}\big](x', v')\end{aligned}$$

where

$$\begin{aligned}\Psi_{2,1,l}(z) &= \int_X \Delta k(z)k(y_2)\varphi_l(z, \bar{w}_1(z, y_2))\Theta(y_2, \hat{w}_1(z, y_2), \bar{w}_2(y_2, x)) \\ &\quad \times E(z, y_2)E(y_2, x)\mathscr{J}(z, y_2, x)m(x, \hat{w}_2(y_2, x))dy_2, \\ \Psi_{2,2,l}(z) &= \int_X \tilde{k}(z)\Delta k(y_2)\varphi_l(z, \bar{w}_1(z, y_2))\Theta(y_2, \hat{w}_1(z, y_2), \bar{w}_2(y_2, x)) \\ &\quad \times E(z, y_2)E(y_2, x)\mathscr{J}(z, y_2, x)m(x, \hat{w}_2(y_2, x))dy_2.\end{aligned}$$

In a similar manner, with $y_1 = \gamma_{(x',v')}(t)$,

$$\begin{aligned}&\Delta\alpha_j(x, x', v') \\ &\quad = \sum_{l=1}^{\infty} \int_0^{\tau_+(x',v')} \mathscr{E}(\vec{\gamma}_{(x',v')}(t)\Theta_l(\vec{\gamma}_{(x',v')}(t)) \\ &\quad \times \int_X \cdots \int_X \left(\sum_{i=1}^{j} \tilde{k}(y_1) \ldots \tilde{k}(y_{i-1})\Delta k(y_i)k(y_{i+1}) \ldots k(y_j) \right) \varphi_l(\gamma_{(x',v')}(t), \bar{w}_1)\end{aligned}$$

$$\times \left(\prod_{i=2}^{j} \Theta(y_i, \hat{w}_{i-1}, \bar{w}_i) E(y_{i-1}, y_i)\right) E(y_j, x) \mathcal{J}(\gamma_{(x',v')}(t), y_2, \ldots, y_j, x)$$
$$\times m(x, \hat{w}_j) dy_j \ldots dy_2 \, dt$$
$$= \sum_{l=1}^{\infty} \sum_{i=1}^{j} I_{\Theta_l \mathscr{E},1} \Psi_{j,i,l}(x', v').$$

Explicitly,

$$\Psi_{j,i,l}(y_1) = \int_X \cdots \int_X \tilde{k}(y_1) \ldots \tilde{k}(y_{i-1}) \Delta k(y_i) k(y_{i+1}) \ldots k(y_j) \varphi_l(y_1, \bar{w}_1)$$
$$\times \left(\prod_{i=2}^{j} \Theta(y_i, \hat{w}_{i-1}, \bar{w}_i) E(y_{i-1}, y_i)\right) E(y_j, x) \mathcal{J}(\cdot) m(x, \hat{w}_j) dy_j \ldots dy_2.$$

It is now necessary to estimate $\|\Psi_{j,i,l}\|_{L^2(X)}$. In what follows, $\|\cdot\|_\infty$ is shorthand for $\|\cdot\|_{L^\infty(Z)}$ where Z is simply the necessary domain of the object for which we are taking the L^∞ norm. Let $X_k = \operatorname{supp} k \cup \operatorname{supp} \tilde{k}$, and $D = \operatorname{dist}(X_k, x) > 0$, and recall that $\max\{\|k\|_\infty, \|\tilde{k}\|_\infty\} < \varepsilon$. First, since $E \leq 1$ and $\|\varphi_l\|_\infty \leq 1$,

$$\|\Psi_{2,1,l}\|_{L^2(X)} \leq \varepsilon \|\Theta\|_\infty \|m\|_\infty \left\| \int_{X_k} \Delta k(z) \mathcal{J}(z, y_2, x) dy_2 \right\|_{L^2(X)}.$$

Since on X_k $d(x, y_2) \geq D$, by Lemma 4.2,

$$\left| \int_{X_k} \Delta k(z) \mathcal{J}(z, y_2, x) dy_1 \right| = |\Delta k(z)| \int_{X_k} J_x(y_2) J_{y_2}(z) dy_2$$
$$\leq |\Delta k(z)| \left(\frac{C_{\kappa_M}}{D}\right)^{n-1} \int_{X_k} J_{y_2}(z) dy_2$$
$$= |\Delta k(z)| \left(\frac{C_{\kappa_M}}{D}\right)^{n-1} (\widetilde{\mathscr{T}} 1)(z)$$

(see (12)). Thus

$$\|\Psi_{2,1,l}\|_{L^2(X)} \leq \varepsilon \|\Theta\|_\infty \|m\|_\infty \left(\frac{C_{\kappa_M}}{D}\right)^{n-1} (\operatorname{diam} X) |\mathbb{S}^{n-1}| \|\Delta k\|_{L^2(X)}.$$

The estimate for $\Psi_{2,2,l}$ is the same although it is derived in a slightly different manner:

$$\|\Psi_{2,2,l}\|_{L^2(X)} \leq \varepsilon \|\Theta\|_\infty \|m\|_\infty \left\| \int_{X_k} \Delta k(y_2) J_x(y_2) J_{y_2}(z) dy_2 \right\|_{L^2(X)}$$
$$\leq \varepsilon \|\Theta\|_\infty \|m\|_\infty \left(\frac{C_{\kappa_M}}{D}\right)^{n-1} \|\widetilde{\mathscr{T}} \Delta k\|_{L^2(X)}$$
$$\leq \varepsilon \|\Theta\|_\infty \|m\|_\infty \left(\frac{C_{\kappa_M}}{D}\right)^{n-1} (\operatorname{diam} X) |\mathbb{S}^{n-1}| \|\Delta k\|_{L^2(X)}.$$

For general j when $i = 1$,

$$\begin{aligned}\|\Psi_{j,1,l}\|_{L^2(X)} &\le \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty \left\|\Delta k(z)\int_{X_k}\cdots\int_{X_k}\mathcal{J}(z,y_2,\ldots,y_j,x)dy_j\ldots dy_2\right\|_{L^2(X)}\\ &\le \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\left(\frac{C_{\kappa_M}}{D}\right)^{n-1}\|\widetilde{\mathcal{T}}^{j-1}1\|_\infty\|\Delta k\|_{L^2(X)}\\ &\le \varepsilon C(\varepsilon\|\Theta\|_\infty(\operatorname{diam} X)|\mathbb{S}^{n-1}|)^{j-2}\|m\|_\infty\left(\frac{C_{\kappa_M}}{D}\right)^{n-1}\|\Delta k\|_{L^2(X)}\end{aligned}$$

where $C = \|\Theta\|_\infty(\operatorname{diam} X)|\mathbb{S}^{n-1}|$. For general j and $i > 1$,

$$\begin{aligned}\|\Psi_{j,i,l}\|_{L^2(X)} &\le \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty \left\|\int_{X_k}\cdots\int_{X_k}\Delta k(y_i)\mathcal{J}(z,y_1,\ldots,y_j,x)dy_j\ldots dy_2\right\|_{L^2(X)}\\ &\le \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\left(\frac{C_{\kappa_M}}{D}\right)^{n-1}\|\widetilde{\mathcal{T}}^{j-i+1}(\Delta k\widetilde{\mathcal{T}}^{i-2}1)\|_{L^2(X)}\\ &\le \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\left(\frac{C_{\kappa_M}}{D}\right)^{n-1}\|\widetilde{\mathcal{T}}\|^{j-i+1}\|\Delta k\|_{L^2(X)}\|\widetilde{\mathcal{T}}^{i-2}1\|_\infty\\ &\le \varepsilon C(\varepsilon\|\Theta\|_\infty(\operatorname{diam} X)|\mathbb{S}^{n-1}|)^{j-2}\|m\|_\infty\left(\frac{C_{\kappa_M}}{D}\right)^{n-1}\|\Delta k\|_{L^2(X)}.\end{aligned}$$

Now, using the relation $\beta I_{\Theta_j\mathscr{E},1} = I_{\Theta_j\mathscr{E},\beta}$, we have

$$\begin{aligned}\left\|I^*_{w,\beta}\beta\sum_{j=2}^{\infty}\Delta\alpha_j\right\|_{H^1(X)} &\le \sum_{j=2}^{\infty}\sum_{i=1}^{j}\sum_{l=1}^{\infty}\|I^*_{w,\beta}I_{\Theta_j\mathscr{E},\beta}\Psi_{j,i,l}\|_{H^1(X)}\\ &\le \sum_{j=2}^{\infty}\sum_{i=1}^{j}\sum_{l=1}^{\infty}\frac{C}{1+l^2}\|\Psi_{j,i,l}\|_{L^2(X)}\\ &\le \sum_{j=2}^{\infty}\sum_{i=1}^{j}\sum_{l=1}^{\infty}\frac{C'}{1+l^2}\varepsilon(\varepsilon\|\Theta\|_\infty(\operatorname{diam} X)|\mathbb{S}^{n-1}|)^{j-2}\|\Delta k\|_{L^2(X)}\\ &\le \varepsilon C''\|\Delta k\|_{L^2(X)}\sum_{j=2}^{\infty} j(\varepsilon\|\Theta\|_\infty(\operatorname{diam} X)|\mathbb{S}^{n-1}|)^{j-2}.\end{aligned}$$

This follows from Proposition 6.2 and Lemma 6.2, with C depending continuously on the C^{3n} norm of g and Θ, and the C^2 norms of σ and m as well. Note that C'' combines C with bounds on $\operatorname{diam} X$ and C_{κ_M}, both of which depend continuously on the metric, certainly so in the C^{3n} norm.

We see that for sufficiently small ε this series converges (thus justifying the formal computations performed above). □

Proof of Theorem 2.2. Fix real analytic (m, g, σ, Θ). Given the hypothesis of Theorem 2.2 we are ensured that both Propositions 6.1 and 6.3 hold. Combining them, we have the existence of a constant C, depending continuously on the C^2

norms of (m, σ), and the C^{3n} norms of (Θ, g) such that

$$\|\Delta k\|_{L^2(X)} \le C\varepsilon \|\Delta k\|_{L^2(X)}$$

and so for $0 \le \varepsilon < C^{-1}$, we must have $k = \tilde{k}$. □

Proof of Lemma 6.2. Given a fixed coordinate system for X (and hence for ΩX), there is a well defined smooth bijection $\theta = \theta_x : \Omega_x X \to \mathbb{S}^{n-1}$, $\theta_x(v) = G(x)^{1/2}v$, where $G(x)^{1/2}$ is the square root of the matrix given by the metric. This change is as smooth as g is. Denoting the inverse of this map by $v(\theta)$ we define $\widetilde{\Theta} \in C^\infty(\Omega X \times \mathbb{S}^{n-1})$

$$\widetilde{\Theta}(x, v', \theta) = \Theta(x, v', v(\theta)).$$

We shall make use of expansions in terms of spherical harmonics. If $f \in H_k$, the space of spherical harmonics of order k, then it can be shown that $\Delta_S f = -k(k + n - 2)f$, where Δ_S is the Laplacian on $\mathbb{S}^{n-1}$. Denote by $Z_k^x(\theta)$ the so-called zonal harmonics for which $f(x) = \langle f, Z_k^x(\theta)\rangle_{L^2(\mathbb{S}^{n-1})}$ for all $f \in H_k$. Then one has (see, for example, [9]),

$$\dim(H_k) = d_k \le c_n(k^{n-2} + 1)$$
$$\|Z_k^x\|_{L^2(\mathbb{S}^{n-1})} = c_n'\sqrt{d_k} \quad \text{for all } x \in \mathbb{S}^{n-1}.$$

Let $\{\tilde{\psi}_{kl}\}_{l=1}^{d_k} \in L^\infty(\mathbb{S}^{n-1})$ be an $L^2(\mathbb{S}^{n-1})$ orthonormal basis for H_k (so $Z_k^x = \sum_{l=1}^{d_k} \tilde{\psi}_{kl}(\theta)\tilde{\psi}_{kl}(x)$). Define $\psi_{kl} := \|\tilde{\psi}_{kl}\|^{-1}_{L^\infty(\mathbb{S}^{n-1})}\tilde{\psi}_{kl}$. Then for each $(x, v') \in \Omega X$,

$$\widetilde{\Theta}(x, v', \theta) = \sum_{k=0}^{\infty}\sum_{l}^{d_k} \Theta_{kl}(x, v')\psi_{kl}(\theta)$$

with

$$\Theta_{kl}(x, v') = \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \widetilde{\Theta}(x, v', \theta)\tilde{\psi}_{kl}(\theta)d\theta.$$

For each θ,

$$|\tilde{\psi}_{kl}(\theta)| = |\langle \tilde{\psi}_{kl}, Z_k^\theta\rangle| \le \|\tilde{\psi}_{kl}\|_{L^2(\mathbb{S}^{n-1})}\|Z_k^\theta\|_{L^2(\mathbb{S}^{n-1})} = c_n'\sqrt{d_k}.$$

Next, since $\tilde{\psi}_{kl} \in H_k$, for any $N \in \mathbb{N}$ and $k \ge 1$,

$$\begin{aligned}
|\Theta_{kl}(x, v')| &= \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})}\left|\int_{\mathbb{S}^{n-1}} \widetilde{\Theta}(x, v', \theta)\tilde{\psi}_{kl}(\theta)d\theta\right| \\
&= \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})}\left|\int_{\mathbb{S}^{n-1}} \widetilde{\Theta}(x, v', \theta)\frac{(\Delta_S)^N\tilde{\psi}_{kl}(\theta)}{(-k(k+n-2))^N}\,d\theta\right| \\
&\le \frac{c_n'\sqrt{d_k}}{(k(k+n-2))^N}\left|\int_{\mathbb{S}^{n-1}} \tilde{\psi}_{kl}(\Delta_S)^N\widetilde{\Theta}(x, v', \theta)d\theta\right| \\
&\le \frac{c_n''k^{(n-2)/2}}{(k(k+n-2))^N}\|(\Delta_S)^N\widetilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})},
\end{aligned}$$

where c''_n depends only on the dimension n. Thus for sufficiently large N (in fact $N \geq 5n/2$), there is $c_{n,N}$ such that

$$|\Theta_{kl}(x, v')| \leq \frac{c_{n,N}}{1 + k^{2n}} \|(\Delta_S)^N \widetilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}.$$

Now renumber the collection of coefficient functions and define new basis functions as follows: with $j = d_0 + d_1 + \cdots + d_{k-1} + l$, set

$$\Theta_j(x, v') = \Theta_{kl}(x, v'), \quad \text{and} \quad \varphi_j(x, v) := \psi_{kl}(\theta_x(v)).$$

Then

$$\Theta(x, v', v) = \sum_{j=1}^{\infty} \Theta_j(x, v')\varphi_j(x, v).$$

Now

$$j \leq \sum_{m=0}^{k} d_m \leq c_n(k+1)(k^{n-2}+1) \leq \hat{c}_n k^n$$

so

$$\begin{aligned} |\Theta_j(x, v')| = |\Theta_{kl}(x, v')| &\leq \frac{c_{n,N}}{1 + k^{2n}} \|(\Delta_S)^N \widetilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})} \\ &\leq \frac{\tilde{c}_{n,N}}{1 + j^2} \|(\Delta_S)^N \widetilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}. \end{aligned}$$

If one applies the same decomposition to $\partial_x^\alpha \Theta(x, v', v)$, $\alpha \in \{0, 1, 2\}^n$, one finds that the coefficients are nothing more than $\partial_x^\alpha \Theta_j(x, v')$ and these satisfy

$$|\partial_x^\alpha \Theta_j(x, v')| \leq \frac{\tilde{c}_{n,N}}{1 + j^2} \|(\Delta_S)^N \partial_x^\alpha \widetilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})} \leq \frac{C}{1 + j^2}.$$

Note that C depends on $2N$ derivatives of $\widetilde{\Theta}$ in the last variable, and hence $2N$ derivatives of g due to the change of variables introduced earlier. This proves the claim of the lemma. □

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