

Boundary Determination of Material Parameters from Electromagnetic Boundary Information*

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Abstract

Consider the following inverse problem: From electromagnetic information obtainable at the boundary of a body, can one determine material parameters, and their normal derivatives, at the boundary of the body? In this paper we answer this question in two physically distinct situations: The first is when the relationship between the electromagnetic fields depends on the conductivity, the electric permittivity and the magnetic permeability of the body, and these parameters together with their normal derivatives are shown to be recoverable at the boundary. The second is when the constitutive relations for the fields are altered so as to further take into account the *chirality* of the body. We also show how a layer stripping algorithm may be derived to estimate the unknown parameters near the boundary in both situations.

The approach is to calculate an explicit asymptotic expansion for the symbol of a boundary operator which is assumed to be known (from boundary measurements); this expansion is shown in each case to determine the unknown parameters at the boundary.

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Introduction

If a body is immersed in a time-dependent electric (or magnetic) field, the induced magnetic (respectively electric) field depends on a number of material parameters of the body. In this work we consider the inverse problem of determining these material parameters at the boundary of the body from knowledge of a certain boundary map. The approach is as in [3] where knowledge of the Dirichlet-to-Neumann map was shown to determine the conductivity and all its derivatives at the boundary of a conducting body subjected to a time-independent electric potential. The result in [3] was obtained for anisotropic conductivities; here we shall restrict ourselves to isotropic parameters.

The paper is divided into two main sections. In section 1 we treat the situation where the body $\Omega \subset \mathbb{R}^3$ has defined on it three material parameters, its conductivity, electric permittivity, and magnetic permeability. In section 2 we consider the situation where in addition the body has a property called *chirality*. Chirality is the existence of a handedness; a chiral molecule is one which cannot be superimposed onto its mirror image, and a chiral material is one which has chiral molecules in its molecular structure (see [4]). One property of a chiral liquid, for example, is that if plane polarized light is passed through a flask of chiral liquid, the plane of polarization may be rotated; the degree of rotation is determined by the relative abundance of “left-handed” versus “right-handed” molecules. The equations determining the relationship between the electric and magnetic fields in this case are Maxwell’s equations with a modification of the constitutive equations describing the electric displacement and the magnetic induction.

We now define the problems more precisely. Take Ω to be a non-chiral body, a smoothly bounded subset of \mathbb{R}^3 . We shall assume that the conductivity γ , the electric permittivity ε , and the magnetic permeability μ are smooth functions satisfying the following conditions:

$$\gamma \geq 0, \quad \varepsilon \geq \varepsilon_0 > 0, \quad \mu \geq \mu_0 > 0$$

throughout Ω . We shall need to make use of the following function spaces: $H^s(\Omega)^3$ consists of three dimensional vector fields whose components are in the usual L^2 -based Sobolev space H^s . Let Div denote the surface divergence on the boundary of Ω , and $\nu(x)$ be the outward unit normal vector at $x \in \partial\Omega$, and define the following space of tangential fields:

$$TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega) = \left\{ F \in (H^{\frac{1}{2}}(\partial\Omega))^3 \mid \nu \cdot F = 0, \text{ and } \text{Div}F \in H^{\frac{1}{2}}(\partial\Omega) \right\}.$$

Suppose that a time-harmonic electric field with fixed frequency ω is applied to Ω and that the tangential component of the induced magnetic field at the boundary of Ω is recorded. The electric field E and magnetic field H are related by Maxwell's equations, which in Euclidean coordinates take the form

$$\nabla \wedge H = (\gamma - i\omega\varepsilon)E, \quad \nabla \wedge E = i\omega\mu H$$

where \wedge denotes the \mathbb{R}^3 vector product. The boundary condition is $\nu \wedge E|_{\partial\Omega} = F$, the tangential component of the applied field at the boundary (see section 1.2). The tangential component of the magnetic field is then $\nu \wedge H$. The boundary map, which we term the *boundary admittance map* following [7], is

$$\Lambda : F = \nu \wedge E|_{\partial\Omega} \mapsto \nu \wedge H|_{\partial\Omega}.$$

The inverse problem being considered for the non-chiral body is then the following: Suppose that we know the map Λ . From this information can we determine the unknown parameters γ , ε and μ at the boundary of Ω ? This question has been answered in the affirmative in [7]. There Somersalo calculates the principal symbol of Λ , and that of the *impedance map* which, when restricted to the Div-spaces, is Λ^{-1} . In the current paper we use an alternate method to calculate not only the principal symbol, but terms of lower orders of homogeneity in an asymptotic expansion for the full symbol of Λ . This approach yields the determination of not only the parameters, but further their normal derivatives at the boundary of Ω . Furthermore, the technique can be applied to the case of chiral media, and in section 2 we prove the analogous boundary determination for a chiral body. In the case of a non-chiral body, *interior* determination of parameters has been shown to be possible from the boundary map, assuming prior knowledge of the parameters near the boundary (see [6]). To remove this assumption in the unique determination of parameters throughout Ω , it suffices to determine the parameters and their first normal derivatives at the boundary; this paper shows that such information is obtainable from Λ .

Remark. A Riccati-type differential equation for Λ can be derived by way of equations (7) and (15) (see sections 1.1 and 1.2), and one may derive a “layer stripping” algorithm to estimate the parameters near the boundary, as was done in [7]. In section 2 we point out the analogous derivation of a Riccati equation for the boundary map in the case of a chiral body.

Section 1 is a treatment of the case of a non-chiral body and the details of the technique are presented. In section 1.1 we rewrite Maxwell's equations in local coordinates as a second order system and derive a factorization by pseudodifferential operators. Section 1.2 expresses the admittance map in terms of these operators, and

in section 1.3 we prove determination of the material parameters by way of calculating the symbol of Λ as a pseudodifferential operator.

Section 2 follows the format of 1, but in the case of a chiral body. The precise formulation of the problem is left to that section.

1 Non-Chiral Media

1.1 A Factorization of Maxwell's Equations

Let Ω be a bounded subset of \mathbb{R}^3 with smooth boundary. Our treatment is local to a point p in the boundary of Ω , and for the moment assume that $p = 0 \in \partial\Omega$, $\Omega \subset \{x_3 > 0\}$ locally, and that $\partial\Omega$ is locally characterized by $x_3 = 0$. We consider a general boundary point at the end of section 1. Let $(E, H) \in \mathcal{D}'(\Omega)^3 \times \mathcal{D}'(\Omega)^3$ be time harmonic electromagnetic fields at frequency ω . In these Euclidean coordinates, such time-harmonic fields are related via the following form of Maxwell's equations:

$$\nabla \wedge H = (\gamma - i\omega\varepsilon)E = -i\omega\sigma E \quad (1)$$

$$\nabla \wedge E = i\omega\mu H \quad (2)$$

Here we use \wedge to denote the \mathbb{R}^3 vector product, and define $i\omega\sigma = i\omega\varepsilon - \gamma$. We also assume that

$$\nabla \cdot (\sigma E) = 0, \quad \nabla \cdot (\mu H) = 0. \quad (3)$$

which physically means that there are no electric charge sources or magnetic poles. Substituting (2) into (1) we obtain

$$-\Delta E + \nabla(\nabla \cdot E) - (\nabla \log \mu) \wedge \nabla \wedge E - \omega^2 \mu \sigma E = 0.$$

But $\nabla \cdot (\sigma E) = 0$, or $\nabla \cdot E = -E \cdot \nabla \log \sigma$, and so

$$-\Delta E - \nabla(E \cdot \nabla \log \sigma) - (\nabla \log \mu) \wedge \nabla \wedge E - \omega^2 \mu \sigma E = 0. \quad (4)$$

We introduce some notation: write $\nabla \log \sigma = (d\sigma_1, d\sigma_2, d\sigma_3)$, $\nabla \log \mu = (d\mu_1, d\mu_2, d\mu_3)$ and let ∂_j denote $\partial/\partial x_j$ and D_{x_j} denote $-i\partial_j$, $j = 1, 2, 3$. Let I denote the 3×3 identity matrix. Then if $x' = (x_1, x_2)$ we may write (4) as

$$\mathcal{M}(x, D)(E) = (D_{x_3}^2 I - \Delta' I - M(x, D_{x'}) - iP(x)D_{x_3} - R(x)) E = 0 \quad (5)$$

with M a 3×3 system of differential operators of order one in x' and depending smoothly on x_3 , and $P(x)$ and $R(x)$ zero order matrix multipliers. Here, Δ' is the two dimensional Laplacian in x_1 and x_2 .

PROPOSITION 1. *There is a pseudodifferential operator $B(x, D_{x'})$ of order one in x' and depending smoothly on x_3 such that*

$$\mathcal{M}(x, D_{x'}) = (D_{x_3}I - iP(x) - iB(x, D_{x'}))(D_{x_3}I + iB(x, D_{x'})) \quad (6)$$

modulo a smoothing operator.

For definitions and properties of pseudodifferential operators see [10].

Proof: We prove the existence of B by explicitly deriving its asymptotic expansion, which is made use of in the sequel. From (5) and (6), we have

$$\Delta' + i[D_{x_3}, B] + M + PB + B^2 + R = 0. \quad (7)$$

Let $B(x, D_{x'}) = (B_{jk})$ have symbol $b(x, \xi') = (b_{jk})$. Here ξ_j is the dual variable to D_{x_j} . Recall that the symbol of $B_{jl}B_{lk}$ is $\sum_{\alpha} \frac{1}{\alpha!} (\partial_{\xi'}^{\alpha} b_{jl})(D_{x'}^{\alpha} b_{lk})$, and so the symbol of B^2 is the matrix

$$\left(\sum_l \sum_{\alpha} \frac{1}{\alpha!} (\partial_{\xi'}^{\alpha} b_{jl})(D_{x'}^{\alpha} b_{lk}) \right)$$

where j and k index the components of the matrix. Similarly one finds that the symbol of $[D_{x_n}, B] = ([D_{x_n}, B_{jk}])$ is

$$\left(\frac{1}{i} \frac{\partial b_{jk}}{\partial x_n} \right).$$

The symbol of M is

$$m(x, \xi') = (m_{jk}) = \begin{pmatrix} i(d\sigma_1\xi_1 - d\mu_2\xi_2) & i(d\sigma_2 + d\mu_2)\xi_1 & i(d\sigma_3 + d\mu_3)\xi_1 \\ i(d\sigma_1 + d\mu_1)\xi_2 & i(d\sigma_2\xi_2 - d\mu_1\xi_1) & i(d\sigma_3 + d\mu_3)\xi_2 \\ 0 & 0 & -i(d\mu_1\xi_1 + d\mu_2\xi_2) \end{pmatrix},$$

and in terms of symbols, (7) becomes

$$-|\xi'|^2\delta_{jk} + \partial_3 b_{jk} + m_{jk} + \sum_l P_{jl}b_{lk} + \sum_l \sum_{\alpha} \frac{1}{\alpha!} (\partial_{\xi'}^{\alpha} b_{jl})(D_{x'}^{\alpha} b_{lk}) + R_{jk} = 0. \quad (8)$$

We write

$$b_{jk} \sim \sum_{q \leq 1} b_{jk}^{(q)}(x, \xi')$$

with $b_{jk}^{(q)}$ homogeneous of degree q in ξ' , and will determine the $b_{jk}^{(q)}$ inductively in q , thus proving the proposition. To this end, the terms in (8) with homogeneity of order two give

$$-|\xi'|^2 \delta_{jk} + \sum_l b_{jl}^{(1)} b_{lk}^{(1)} = -|\xi'|^2 I + (b_{jk}^{(1)})^2 = 0$$

and we choose the solution $(b_{jk}^{(1)}) = -|\xi'|I$. Next the terms homogeneous of order one, together with $b_{jk}^{(1)} = -|\xi'| \delta_{jk}$ give

$$\begin{aligned} 0 &= \partial_3 b_{jk}^{(1)} + m_{jk} + \sum_l P_{jl} b_{lk}^{(1)} + \sum_l (b_{jl}^{(1)} b_{lk}^{(0)} + b_{jl}^{(0)} b_{lk}^{(1)}) + \sum_{|\alpha|=1} \partial_{\xi'}^\alpha b_{jl}^{(1)} D_{x'}^\alpha b_{lk}^{(1)} \\ &= m_{jk} - P_{jk} |\xi'| - 2|\xi'| b_{jk}^{(0)}. \end{aligned}$$

Since we are solving only modulo smoothing, we put

$$b_{jk}^{(0)} = \frac{m_{jk} - P_{jk} |\xi'|}{2|\xi'|}. \quad (9)$$

Continuing inductively, if $q < 0$ (the case $q = 0$ is similar), the terms homogeneous of order q in (8) are

$$0 = \partial_3 b_{jk}^{(q)} + \sum_l P_{jl} b_{lk}^{(q)} + \sum_l \sum_{0 \leq |\alpha| \leq -q+2} \frac{1}{\alpha!} \sum_{q+|\alpha|-1 \leq s \leq 1} \partial_{\xi'}^\alpha b_{jl}^{(s)} D_{x'}^\alpha b_{lk}^{(q+|\alpha|-s)}.$$

The only terms involving $b_{jk}^{(q-1)}$ are when $|\alpha| = 0$ and $s = 1$ or $s = q-1$ in which case

$$\sum_l (b_{jl}^{(1)} b_{lk}^{(q-1)} + b_{jl}^{(q-1)} b_{lk}^{(1)}) = -2|\xi'| b_{jk}^{(q-1)}$$

and thus we set $b_{jk}^{(q-1)} =$

$$\frac{1}{2|\xi'|} \left(\partial_3 b_{jk}^{(q)} + \sum_l P_{jl} b_{lk}^{(q)} - \sum_l \sum_{0 \leq |\alpha| \leq -q+2} \frac{1}{\alpha!} \sum_{\substack{q+|\alpha|-1 \leq s \leq 1 \\ s \neq 1, q-1 \text{ when } |\alpha|=0}} \partial_{\xi'}^\alpha b_{jl}^{(s)} D_{x'}^\alpha b_{lk}^{(q+|\alpha|-s)} \right).$$

□

1.2 The Boundary Admittance Map, Λ

It was shown in [8] that for all but a discrete set of $\omega > 0$, with no accumulation points, the following Dirichlet problem has a unique solution: for $F \in TH_{\text{Div}}^{\frac{1}{2}}(\partial\Omega)$, let $(E, H) \in \mathcal{D}'(\Omega)^3 \times \mathcal{D}'(\Omega)^3$ be the solution to

$$\begin{aligned} \nabla \wedge H &= -i\omega\sigma E \\ \nabla \wedge E &= i\omega\mu H \quad \nu \wedge E|_{\partial\Omega} = F \end{aligned} \quad (10)$$

where ν is the outward unit normal to the boundary of Ω .

PROPOSITION 2. *If E solves (10), then*

$$\left. \frac{\partial}{\partial x_3} E \right|_{\partial\Omega} = BE|_{\partial\Omega}. \quad (11)$$

modulo a smoothing operator.

Proof: Certainly, $\mathcal{M}(x, D)E = 0$. Let (x', x_3) be local coordinates, for $x_3 \in [0, T]$. Since the principal symbol of $\mathcal{M}(x, D)$ is $-|\xi|^2 I$, the plane $\{x_3 = 0\}$ is non-characteristic, and so \mathcal{M} is partially hypoelliptic with respect to this boundary (see [2] p107). Thus since E solves (10), E is smooth in the normal direction; that is, $E \in C^\infty([0, T]; \mathcal{D}(\mathbb{R}^2))^3$ locally. By Proposition (1), (10) is locally equivalent to

$$(D_{x_3} I + iB)E = U, \quad \nu \wedge E|_{x_3=0} = F, \quad (12)$$

$$(D_{x_3} I - iP - iB)U = W \in C^\infty([0, T] \times \mathbb{R}^2)^3 \quad (13)$$

with E and U in $C^\infty([0, T]; \mathcal{D}(\mathbb{R}^2))^3$ (again, $(D_{x_3} I + iB)$ is hypoelliptic with respect to the boundary). We may view (13) as a backwards generalized heat equation; indeed with $t = T - x_n$, we have

$$\frac{\partial U}{\partial t} - (B + P)U = -iW. \quad (14)$$

By interior regularity for $\mathcal{M}(x, D)$, E is smooth in the interior of Ω , and hence so is U ; in particular, $U|_{x_3=T}$ is smooth. Now the principal symbol of B is $b^1 = -|\xi'|I$ and so the solution operator for (14) is smoothing for $t > 0$ (see [9] p134), and

$$(D_{x_3} I + iB)E = U \in C^\infty([0, T] \times \mathbb{R}^2)^3$$

locally. In particular,

$$D_{x_3} E|_{x_3=0} = -i BE|_{x_3=0} + U|_{x_3=0}$$

which completes the proof since $U|_{x_3=0}$ is smooth. \square

Recall that if (E, H) solves (10) then the admittance map $\Lambda = \Lambda(\gamma, \varepsilon, \mu)$ is the map $\Lambda : \nu \wedge E|_{\partial\Omega} \mapsto \nu \wedge H|_{\partial\Omega}$ where $\nu = (0, 0, -1)$ is the outward unit normal to the boundary $\partial\Omega$. From (2),

$$\left(\begin{array}{c} E_2 \\ -E_1 \\ 0 \end{array} \right) \Big|_{x_3=0} \xrightarrow{\Lambda} \frac{1}{i\omega\mu} \left(\begin{array}{c} \partial_3 E_1 - \partial_1 E_3 \\ \partial_3 E_2 - \partial_2 E_3 \\ 0 \end{array} \right) \Big|_{x_3=0}.$$

By Proposition 2,

$$\partial_3 E_j = B_{j1}E_1 + B_{j2}E_2 + B_{j3}E_3, \quad j = 1, 2, 3,$$

and from (3)

$$\partial_3 \sigma E_3 + \sigma \partial_3 E_3 = -(\sigma \partial_1 + \partial_1 \sigma)E_1 - (\sigma \partial_2 + \partial_2 \sigma)E_2;$$

combining these, we have

$$(\partial_3 \sigma + \sigma B_{33})E_3 = -(\sigma B_{31} + \sigma \partial_1 + \partial_1 \sigma)E_1 - (\sigma B_{32} + \sigma \partial_2 + \partial_2 \sigma)E_2.$$

Let $J(x, D_{x'}) = (\partial_3 \sigma + \sigma B_{33})$, and $K(x, D_{x'})$ be a pseudodifferential operator of order -1 in x' such that the composition KJ is the identity modulo smoothing. Then Λ is given by the 2×2 system with components for $j = 1, 2$

$$\begin{aligned} \Lambda_{j1} &= \frac{1}{i\omega\mu} ((\partial_j K - B_{j3}K)(\sigma B_{32} + \sigma \partial_2 + \partial_2 \sigma) + B_{j2}) \\ \Lambda_{j2} &= \frac{1}{i\omega\mu} ((-\partial_j K + B_{j3}K)(\sigma B_{31} + \sigma \partial_1 + \partial_1 \sigma) - B_{j1}). \end{aligned} \quad (15)$$

We write the symbol of J as $j(x, \xi') \sim \sum_{l \leq 1} j_l(x, \xi')$ with j_l homogeneous of degree l in ξ' and given by

$$j_1 = -\sigma|\xi'|, \quad j_0 = \partial_3 \sigma + \sigma b_{33}^{(0)}, \quad j_l = \sigma b_{33}^{(l)}, \quad l < 0.$$

To compute the symbol k of K modulo $S^{-\infty}$, we write $k \sim \sum_{r \leq -1} k_r(x, \xi')$ and consider the terms of decreasing homogeneity in the identity

$$1 = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} (\partial_{\xi'}^\alpha k(x, \xi')) (D_{x'}^\alpha j(x, \xi')) = \text{the symbol of } KJ.$$

1.3 The Symbol of Λ and Boundary Determination of μ , ε , γ

In this section we show that knowledge of the boundary admittance map Λ is sufficient to determine the unknown parameters ε , γ and μ on the boundary of Ω . Let

$$\lambda(x', \xi') = (\lambda_{jk}(x', \xi')) \sim \left(\sum_{q \leq 1} \lambda_{jk}^{(q)} \right)$$

be the symbol of Λ . Since we are assuming complete knowledge of Λ , we know the full symbol $\lambda(x', \xi')$ for all $(x', \xi') \in \mathbb{R}^2 \times \mathbb{R}^2$.

First, calculating the terms of homogeneity one in (15), one finds that the principal symbol of Λ is

$$\lambda^{(1)}(x', \xi') = \frac{-1}{i\omega\mu(x', 0)|\xi'|} \begin{pmatrix} -\xi_1\xi_2 & -\xi_2^2 \\ \xi_1^2 & \xi_1\xi_2 \end{pmatrix}.$$

Thus the principal symbol of Λ determines μ on the boundary. Note that $\lambda^{(1)}$ then also determines the tangential derivatives of μ on the boundary. We remark that this is the negative of the principal symbol calculated in [7], the reason for this being that in [7] ξ_j is taken to be the dual variable to $i\partial_j = -D_{x_j}$.

Next, we calculate the terms of homogeneity zero in (15). For example,

$$i\omega\mu\lambda_{11}^{(0)} = i\xi_1\sigma k_{-1}b_{32}^{(0)} - \xi_1\xi_2\sigma k_{-2} - \xi_1\xi_2 \sum_{|\alpha|=1} \partial_{\xi'}^{\alpha} k_{-1} D_x^{\alpha} \sigma + i\xi_1\sigma k_{-1} d\sigma_2 + b_{12}^{(0)} - i\xi_2 b_{13}^{(0)} \sigma k_{-1}.$$

The symbols $b_{32}^{(0)}$, $b_{12}^{(0)}$ and $b_{13}^{(0)}$ are given by (9), and (1.2) yields

$$k_{-1} = \frac{-1}{\sigma|\xi'|}, \quad k_{-2} = \frac{i}{2\sigma|\xi'|^3} (id\sigma_1|\xi'| + \xi_1 d\mu_1 + \xi_2 d\mu_2 + 2\xi_1 d\sigma_1 + 2\xi_2 d\sigma_2)$$

which gives

$$i\omega\mu\lambda_{11}^{(0)}(x', \xi') = \frac{i\xi_1}{2|\xi'|^3} (2|\xi'|^2 d\mu_2 - \xi_1\xi_2 d\mu_1 - \xi_2^2 d\mu_2 + i\xi_2|\xi'| d\mu_3).$$

Since we have determined μ , and hence $d\mu_1$ and $d\mu_2$, on the boundary, $\lambda_{11}^{(0)}$ determines $d\mu_3$ there; this is the only new information to be gained from all four of the components of the matrix $\lambda^{(0)}$.

Remark. A similar analysis of the impedance map Λ^{-1} shows that the two highest terms of homogeneity in the expansion of the symbol for Λ^{-1} determine σ and its normal derivative at the boundary.

To determine higher order normal derivatives of the parameters at the boundary, it is necessary to continue and calculate the components of $\lambda^{(-1)}(x', \xi')$. The calculation is involved, but is straightforward in manner. To extract the desired information we choose to set ξ' equal to $(1,0)$ and then $(0,1)$. In what follows, $f_j(x')$ are *known* functions of x' (each is defined by functions already determined). We find:

$$2i\omega\mu\lambda_{11}^{(-1)}(x', (1,0)) - f_1 = d\mu_1 d\sigma_2 - \partial_1 d\sigma_2 \quad (16)$$

$$2i\omega\mu\lambda_{11}^{(-1)}(x', (0,1)) - f_2 = -\partial_1 d\sigma_2 \quad (17)$$

$$2i\omega\mu\lambda_{12}^{(-1)}(x', (1,0)) - f_3 = -d\mu_1 d\sigma_1 - (1+i)(2d\sigma_1^2 - d\sigma_2^2 - \partial_2 d\sigma_2) \\ - (1-2i)\partial_1 d\sigma_1 - 2\omega^2\mu\sigma \quad (18)$$

$$4i\omega\mu\lambda_{12}^{(-1)}(x', (0,1)) - f_4 = -\partial_3 d\mu_3 + 2\partial_1 d\sigma_1 - 2\omega^2\mu\sigma \quad (19)$$

$$4i\omega\mu\lambda_{21}^{(-1)}(x', (1,0)) - f_5 = \partial_3 d\mu_3 - 2\partial_2 d\sigma_2 + 2\omega^2\mu\sigma$$

$$2i\omega\mu\lambda_{21}^{(-1)}(x', (0,1)) - f_6 = d\mu_2 d\sigma_2 + (1+i)(2d\sigma_2^2 - d\sigma_1^2 - \partial_1 d\sigma_1) \\ + (1+2i)\partial_2 d\sigma_2 + 2\omega^2\mu\sigma$$

$$2i\omega\mu\lambda_{22}^{(-1)}(x', (1,0)) - f_7 = \partial_2 d\sigma_1 \quad (20)$$

$$2i\omega\mu\lambda_{22}^{(-1)}(x', (0,1)) - f_8 = -d\mu_2 d\sigma_1 + \partial_2 d\sigma_1 \quad (21)$$

It is easy to see that equations (16) to (21) determine the unknown parameters on the boundary: (16) and (17) determine $d\sigma_2$, and (20) and (21) determine $d\sigma_1$; furthermore, the tangential derivatives of these functions are known. With this additional knowledge, (18) gives σ , and hence ε and γ , and then (19) gives $\partial_3 d\mu_3$.

Let us summarize these findings: recalling the definitions of $d\sigma_j$ and $d\mu_j$, the highest three terms of homogeneity in the asymptotic expansion of λ determine ε , γ , μ and $\partial_\nu\mu$ on $\partial\Omega$, where ∂_ν denotes the normal derivative. Since these functions are known on the boundary, we remark that their tangential derivatives are also known there. We remark further that $\lambda^{(-1)}$ does not determine the normal derivatives of ε and γ ; it seems reasonable to expect $\lambda^{(-2)}$ to determine these, however the calculations become considerably more cumbersome. In fact we can expect λ to determine the derivatives of all the parameters, of all orders, at the boundary.

We now consider the situation where $\partial\Omega$ is not flat near p . In local coordinates near p , if $E = \sum_j E_j \frac{\partial}{\partial x_j}$, denote by E^b the one-form $E^b = \sum_j E_j dx^j$ obtained via

duality. Then Maxwell's equations take the form

$$(*dE^b)^\# = i\omega\mu H, \quad (*dH^b)^\# = -i\omega\sigma E$$

where $*$ is the Hodge-star operator for the metric induced from the Euclidean metric in \mathbb{R}^3 , and $\#$ reinterprets the one-form as a vector field via duality. We choose local coordinates near p to be boundary normal coordinates (see [3] p1101) with the coordinates for $\partial\Omega$ being Riemann normal coordinates. Then $\partial\Omega$ is locally characterized by $x_3 = 0$ and the induced metric is Euclidean at p and has all first derivatives vanishing at p . With this choice the calculations reduce to those of the flat case for the two highest order terms in the expansion of Λ . Thus by considering Λ and Λ^{-1} , the parameters and their first normal derivatives are obtainable at p .

2 Chiral Media

2.1 The Modified Maxwell's Equations

Suppose now that Ω is a body with chirality described by a smooth function β . In this section we wish to apply the techniques of the previous section to determine β together with μ , ε and γ on the boundary of Ω . The same arguments of section 1 regarding a general point on the boundary of Ω versus the case of a flat boundary apply here and so we take $\partial\Omega$ to be $\{x_3 = 0\}$ near 0. If D is the electric displacement and B is the magnetic induction, Maxwell's equations read

$$\nabla \wedge H = -i\omega D \quad (22)$$

$$\nabla \wedge E = i\omega B \quad (23)$$

where, for a Chiral body, B and D are related to E and H through the constitutive equations (see [1], [5])

$$B = \mu H + \beta E, \quad D = \sigma E - \beta H \quad (24)$$

(as before $i\omega\sigma = i\omega\varepsilon - \gamma$). We assume the displacement and induction to be divergence free,

$$\nabla \cdot B = 0, \quad \nabla \cdot D = 0. \quad (25)$$

The system employed here is not exactly the same as that of section 1. It is no longer possible to use the divergence free conditions to decouple the system and find the

analogue of equation (5) in terms of E alone. Instead we consider the electric and magnetic fields together. Taking the curl of (22) and (23), we may write

$$-\Delta \begin{pmatrix} E \\ H \end{pmatrix} + \nabla \begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} + i\omega \begin{pmatrix} -\nabla\mu \wedge H \\ \nabla\sigma \wedge E \end{pmatrix} = Z \begin{pmatrix} E \\ H \end{pmatrix} \quad (26)$$

where Z is a zero order matrix multiplier. The conditions (25) imply

$$\begin{pmatrix} \nabla \cdot E \\ \nabla \cdot H \end{pmatrix} = \frac{1}{\sigma\mu + \beta^2} \begin{pmatrix} (-\beta\nabla\beta - \mu\nabla\sigma) \cdot E + (\mu\nabla\beta - \beta\nabla\mu) \cdot H \\ (\beta\nabla\sigma - \sigma\nabla\beta) \cdot E + (-\beta\nabla\beta - \sigma\nabla\mu) \cdot H \end{pmatrix}.$$

Further, from (22) and (23),

$$i\omega \begin{pmatrix} -\nabla\mu \wedge H \\ \nabla\sigma \wedge E \end{pmatrix} = \frac{1}{\sigma\mu + \beta^2} \begin{pmatrix} -\sigma\nabla\mu \wedge & -\beta\nabla\mu \wedge \\ \beta\nabla\sigma \wedge & -\mu\nabla\sigma \wedge \end{pmatrix} \begin{pmatrix} \nabla \wedge E \\ \nabla \wedge H \end{pmatrix}$$

Combining these with (26), we obtain an equation of the form

$$\mathcal{N}(x, D) \begin{pmatrix} E \\ H \end{pmatrix} = (D_{x_3}^2 - \Delta' - N(x, D_{x'}) - iQ(x)D_{x_3} - S(x)) \begin{pmatrix} E \\ H \end{pmatrix} = 0 \quad (27)$$

where $N(x, D)$ is a 6×6 system of differential operators of order one in x' and depending smoothly on x_3 , and $Q(x)$ and $S(x)$ are zero order matrix multipliers. As in Proposition 1, there is a pseudodifferential operator $C(x, D_{x'})$ of order one in x' and depending smoothly on x_3 factorizing \mathcal{N} as

$$\mathcal{N}(x, D) = (D_{x_3} - iQ(x) - iC(x, D_{x'}))(D_{x_3} + iC(x, D_{x'})) \quad (28)$$

modulo smoothing.

2.2 The Admittance Map Π for Chiral Ω

Let $(E, H) \in \mathcal{D}'(\Omega)^3 \times \mathcal{D}'(\Omega)^3$ solve

$$\begin{aligned} \nabla \wedge E &= i\omega(\mu H + \beta E) \\ \nabla \wedge H &= i\omega(\sigma E - \beta H). \end{aligned}$$

With the analogous proof of Proposition 2 we have for such solutions (E, H) ,

$$\frac{\partial}{\partial x_3} \begin{pmatrix} E \\ H \end{pmatrix} \Big|_{\partial\Omega} = C \begin{pmatrix} E \\ H \end{pmatrix} \Big|_{\partial\Omega} \quad (29)$$

modulo smoothing. From (25) and (29), on $\partial\Omega$,

$$\left[\begin{pmatrix} \partial_3\beta & \partial_3\mu \\ \partial_3\sigma & -\partial_3\beta \end{pmatrix} + \begin{pmatrix} \beta & \mu \\ \sigma & -\beta \end{pmatrix} \begin{pmatrix} C_{33} & C_{36} \\ C_{63} & C_{66} \end{pmatrix} \right] \begin{pmatrix} E_3 \\ H_3 \end{pmatrix} = \begin{pmatrix} -(\partial_1\beta + \beta\partial_1 + \beta C_{31} + \mu C_{61})E_1 - (\partial_2\beta + \beta\partial_2 + \beta C_{32} + \mu C_{62})E_2 \\ -(\partial_1\mu + \mu\partial_1 + \beta C_{34} + \mu C_{64})H_1 - (\partial_2\mu + \mu\partial_2 + \beta C_{35} + \mu C_{65})H_2 \\ -(\partial_1\sigma + \sigma\partial_1 + \sigma C_{31} - \beta C_{61})E_1 - (\partial_2\sigma + \sigma\partial_2 + \sigma C_{32} - \beta C_{62})E_2 \\ +(\partial_1\beta + \beta\partial_1 - \sigma C_{34} + \beta C_{64})H_1 + (\partial_2\beta + \beta\partial_2 - \sigma C_{35} + \beta C_{65})H_2 \end{pmatrix}$$

or

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} E_3 \\ H_3 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad \text{say}$$

Let G be a pseudodifferential operator of order -1 such that $GF = Id$ modulo smoothing. We shall use $\Pi = \Pi(\beta, \varepsilon, \mu, \gamma)$ to denote the admittance map for a chiral body; using (22) and (23) this is the map

$$\begin{pmatrix} E_2 \\ -E_1 \\ H_2 \\ -H_1 \end{pmatrix} \Big|_{x_3=0} \xrightarrow{\Pi} \begin{pmatrix} \frac{1}{i\omega\mu}(\partial_3E_1 - \partial_1E_3) \\ \frac{1}{i\omega\mu}(\partial_3E_2 - \partial_2E_3) \\ \frac{-1}{i\omega\sigma}(\partial_3H_1 - \partial_1H_3) \\ \frac{1}{i\omega\sigma}(\partial_3H_2 - \partial_2H_3) \end{pmatrix} \Big|_{x_3=0} + \begin{pmatrix} -\frac{\beta}{\mu}E_2 \\ \frac{\beta}{\mu}E_1 \\ -\frac{\beta}{\sigma}H_2 \\ \frac{\beta}{\sigma}H_1 \end{pmatrix} \Big|_{x_3=0}.$$

The components of the 4×4 system Π can be computed in terms of C and G , for example,

$$\begin{aligned} \Pi_{11} &= \frac{1}{i\omega\mu} \{ C_{12} - (C_{13} + C_{16} - \partial_1)[G_{11}(\partial_2\beta + \beta\partial_2 + \beta C_{32} + \mu C_{62}) \\ &\quad + G_{12}(\partial_2\sigma + \sigma\partial_2 + \sigma C_{32} - \beta C_{62})] \} - \frac{\beta}{\mu} \\ \Pi_{12} &= \frac{-1}{i\omega\mu} \{ C_{11} - (C_{13} + C_{16} - \partial_1)[G_{11}(\partial_1\beta + \beta\partial_1 + \beta C_{32} + \mu C_{62}) \\ &\quad + G_{12}(\partial_1\sigma + \sigma\partial_1 + \sigma C_{32} - \beta C_{62})] \} \\ &\dots \\ \Pi_{31} &= \frac{-1}{i\omega\sigma} \{ C_{42} - (C_{43} + C_{46} - \partial_1)[G_{21}(\partial_2\beta + \beta\partial_2 + \beta C_{32} + \mu C_{62}) \\ &\quad + G_{22}(\partial_2\sigma + \sigma\partial_2 + \sigma C_{32} - \beta C_{62})] \} \\ \Pi_{32} &= \frac{1}{i\omega\sigma} \{ C_{41} - (C_{43} + C_{46} - \partial_1)[G_{21}(\partial_1\beta + \beta\partial_1 + \beta C_{32} + \mu C_{62}) \\ &\quad + G_{22}(\partial_1\sigma + \sigma\partial_1 + \sigma C_{32} - \beta C_{62})] \} \end{aligned} \tag{30}$$

2.3 Boundary Determination of μ , ε , γ and β

Writing the symbol of Π as $\pi(x', \xi') = \pi_{jk}(x', \xi') \sim \sum_{q \leq 1} \pi_{jk}^{(q)}$, the principal symbol of Π is computed to be

$$\pi^{(1)}(x', \xi') = \frac{1}{i\omega|\xi'|} \begin{bmatrix} \frac{1}{\mu} \begin{pmatrix} \xi_1 \xi_2 & \xi_2^2 \\ -\xi_1^2 & -\xi_1 \xi_2 \end{pmatrix} & 0 \\ 0 & \frac{1}{\sigma} \begin{pmatrix} -\xi_1 \xi_2 & -\xi_2^2 \\ \xi_1^2 & \xi_1 \xi_2 \end{pmatrix} \end{bmatrix}$$

Clearly this determines μ and σ (and hence ε and γ) at the boundary. The computations for $\pi^{(0)}$ are analogous to those for λ . We outline the arguments leading to the determination of the unknown parameters. In what follows, $g_j(x)$ are known functions at each stage, and ξ' is chosen as appropriate.

$$\begin{aligned} \pi_{13}^{(0)}(x', (1, 0)) - g_1(x') &= \frac{-\beta}{\sigma\mu + \beta^2} \\ \pi_{14}^{(0)}(x', (0, 1)) - g_2(x') &= \left(\frac{\partial_2 \mu + i\partial_3 \mu}{2\omega\mu} \right) \frac{-\beta}{\sigma\mu + \beta^2} \end{aligned} \quad (31)$$

and so these combine to determine $\partial_3 \mu$. Next,

$$\pi_{12}^{(0)}(x', (0, 1)) - g_3(x') = \left(\frac{\partial_2 \mu + i\partial_3 \mu}{2\omega} \right) \frac{1}{\sigma\mu + \beta^2}$$

determines $\sigma\mu + \beta^2$ which, with (31), gives β . Finally

$$\pi_{11}^{(0)}(x', (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})) - g_4(x') = \frac{i\partial_3 \beta}{2\omega(\sigma\mu + \beta^2)}$$

determines $\partial_3 \beta$.

To summarize, the two highest order terms in the expansion of π determine β , μ , ε and γ , and the normal derivatives of each on $\partial\Omega$.

2.4 Remark: Layer Stripping for Chiral Bodies

For a non-chiral body, a *layer stripping* algorithm was proposed in [7] to estimate the parameters near the boundary. This algorithm essentially proceeds as follows: from

the boundary map Λ , the parameters are determined at the boundary. One then *strips* away the known “surface layer” to expose a new boundary, and by way of a Riccati-type equation for Λ , an approximate boundary map for the new boundary is obtained. From this the parameters are estimated a little below the surface and the procedure can be repeated. The essential component of this algorithm is the existence of a Riccati-type equation with which to advance the boundary map in from the original surface. From (27) and (28), together with the fact that $D_{x_3}f(x', 0) = 0$, we have the following Riccati-type equation for C :

$$\partial_3 C = -C^2 - QC - \Delta' - N - S .$$

Now we may calculate the normal derivative of Π by way of (30), in terms of operators which are determined from the symbol of Π . This yields a Riccati-type equation for Π , and layer stripping may be applied in the case of a chiral body.

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References

- [1] Baum and Kritikos, 1995, *Electromagnetic Symmetry* (Washington D.C.: Taylor and Francis)
- [2] Hörmander L., 1964, *Linear Partial Differential Operators* (Berlin: Springer-Verlag)
- [3] Lee J. and Uhlmann G., 1989, Determining Anisotropic Real-Analytic Conductivities by Boundary Measurements *Comm. Pure Appl. Math.* **42** 1097-1112
- [4] Graham Solomons T. W., 1992, *Organic Chemistry* (New York: John Wiley and Sons Inc.)
- [5] Lakhtakia A., Varadan V. K. and Varadan V. V., 1989, *Time-Harmonic Electromagnetic Fields in Chiral Media*, Lecture Notes in Physics, Vol 335 (Berlin: Springer-Verlag)
- [6] Ola P., Päivärinta L. and Somersalo E., 1993, An Inverse Boundary Problem in Electrodynamics *Duke Math. J.* **70** 617-653

- [7] Somersalo E., 1994, Layer Stripping for Time Harmonic Maxwell's Equations with High Frequency *Inverse Problems*, Vol. 10 No. 2, 449-466
- [8] Somersalo E., Isaacson D. and Cheney M., 1992, A Linearized Inverse Boundary Value Problem for Maxwell's Equations *J. Comput. Appl. Math.* **42** 123-136
- [9] Treves F., 1980, *Introduction to Pseudodifferential and Fourier Integral Operators, Vol. I* (New York: Plenum Press)
- [10] Taylor, M. E., 1981, *Pseudodifferential Operators* (Princeton: Princeton University Press)