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Optical Tomography on Simple Riemannian Surfaces

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Optical tomography means the use of near-infrared light to determine the optical absorption and scattering properties of a medium. In the stationary Euclidean case the dynamics are modeled by the radiative transport equation, which assumes that, in the absence of interaction, particles follow straight lines. Here we shall study the problem in the presence of a Riemannian metric where particles follow the geodesic flow of the metric. In particular, we study the problem in dimension two, where the analysis is more delicate than in the higher dimensional cases.

Keywords Boltzmann equation; Riemannian metric; Transport equation.

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1. Introduction

Optical tomography means the use of near-infrared light to determine the optical absorption and scattering properties of a medium. One prescribes a distribution of particles (photons in this case) entering the body at its boundary and measures the resulting flux of particles leaving the body. One then seeks to determine the absorption and scattering properties interior to the medium from knowledge of the "albedo" operator, the map from the incoming to the outgoing distributions of particles. Optical tomography is beginning to be applied to the problems of medical imaging; see Arridge (1999) for a review article. In the stationary Euclidean case, the dynamics are modeled by the radiative transport equation, which assumes that, in the absence of interaction, particles follow straight lines. We are concerned here with the situation of particles moving in an ambient field represented by a Riemannian metric. The consequence is that in the absence of interaction a particle will now follow the geodesics of the metric.

We first describe the problem in some generality. Let M be a bounded open domain in $\mathbb{R}^n$ with smooth boundary, and let g be a Riemannian metric on M. If $f(x, v)$ represents the density of particles at position x with velocity vector v in the

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unit tangent sphere at x, $\Omega_x M$, then the stationary linear transport equation is

$$-\mathcal{D}f(x,v) - \sigma_a(x,v)f(x,v) + \int_{\Omega_x M} k(x,v',v)f(x,v')dv'_x = 0. \tag{1}$$

The operator $\mathcal{D}$ is the derivative along the geodesic flow (see (3) below) which in the case of g being Euclidean is simply $\mathcal{D}f(x,v) = v \cdot \nabla_x f(x,v)$. The coefficient σ_a is the absorption coefficient, and k is the scattering kernel; σ_a describes the probability of a particle with position x and velocity v being absorbed, and k describes that of a particle with position x and velocity v' being "scattered" to velocity v. We restrict ourselves to the case where all particles travel at unit speed and hence use the unit sphere bundle ΩM rather than the full tangent bundle TM. The measure dv'_x in (1) is the Euclidean volume form on the tangent sphere $\Omega_x M$ determined by the metric g at x (see definition 2 below). Define the incoming and outgoing bundles

$$\Gamma_\pm = \{(x,v) \in \Omega M : x \in \partial M, \ \pm\langle v, \nu\rangle > 0\}$$

on the boundary ∂M of M, where ν is the outward unit normal vector to ∂M and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{g_x}$ is the inner product with respect to the metric g at x. If f_- is a distribution of particles defined on Γ_-, let f be the solution to (1), should it exist, with boundary condition $f|_{\Gamma_-} = f_-$. Then the albedo operator is the map $\mathcal{A}$ which maps f_- to the outgoing flux of particles:

$$\mathcal{A}f_- = f|_{\Gamma_+}.$$

The inverse problem, the problem of optical tomography, is to determine uniquely σ_a and k from knowledge of $\mathcal{A}$.

When the ambient metric is Euclidean, the inverse problem described has been well studied. Choulli and Stefanov (1999) show that the most singular part of the distributional kernel of the albedo operator determines the X-ray transform of the absorption coefficient and that the next most singular part determines the collision kernel k in dimensions three and greater. Their method fails to determine k from $\mathcal{A}$ in the two-dimensional case. Stefanov and Uhlmann (2003) performed a more precise analysis of the second term in the singular expansion of the distributional kernel to obtain a unique determination of k in dimension two, under the assumption that k is small relative to σ_a, with an explicit constant given. They also give stability estimates. Also in dimension two, Tamasan (2002) shows a determination of k that is homogeneous in x, and Tamasan (2003) proves a determination of a kernel that is "weakly anisotropic." The time-dependent analog of (1) and its inverse problem were considered in Choulli and Stefanov (1996). Stability estimates have also been obtained in Romanov (1996) and Wang (1999).

In the Riemannian setting, an inverse source problem for the stationary transport equation is addressed in Sharafutdinov (1999) (see also Sharafutdinov, 1994) where $k(x,v,v') = k(x,\langle v,v'\rangle)$ is assumed to depend on the angle between v and v' and the object of interest is the reconstruction of an isotropic source term. In Ferwerda (1999) the (time dependent) radiative transfer equation is derived for a medium with spatially varying refractive index and with scattering kernel k independent of position. Such a refractive index is represented by a Riemannian metric. In McDowall (2004) the author proved the analogous result to that of Choulli and Stefanov (1999), namely that in the presence of a Riemannian metric

$\mathcal{A}$ uniquely determines σ_a and k in dimensions three and greater, and only σ_a in dimension two.

In this article we address the two-dimensional case in the presence of a Riemannian metric, with a smallness assumption on an otherwise general k. One may understand such smallness in k as the body being "nearly transparent." Let M be a two-dimensional Riemannian manifold with smooth, strictly convex boundary. We make the following assumptions on the geometry of (M, g):

M1. The sectional curvature of (M, g) is bounded above by κ_0.
M2. If $\kappa_0 > 0$ we assume that (M, g) has no *focal points*, that is, for every geodesic $\gamma : [a, b] \to M$ and every nonzero Jacobi field $J(t)$ along γ satisfying $J(a) = 0$, we have $\|J(t)\|$ as a strictly increasing function on $[a, b]$. Note that if $\kappa_0 \leq 0$ then (M, g) necessarily has no focal points.
M3. In the case that $\kappa_0 > 0$ we assume that the diameter A of (M, g) satisfies $A < \pi/(2\sqrt{\kappa_0})$. There is no restriction on the diameter of (M, g) when $\kappa_0 \leq 0$, other than it is finite.

It follows that (M, g) is "simple." In particular, for any $x \in \overline{M}$ the exponential map $\exp_x : \exp_x^{-1}(\overline{M}) \to \overline{M}$ is a diffeomorphism. Consequently M is diffeomorphic to a disk. We make the following assumptions on (σ_a, k):

A1. Even in the Euclidean case and when $k = 0$, $\mathcal{A}$ does not uniquely determine σ_a (see Choulli and Stefanov, 1999; Stefanov and Uhlmann, 2003) and so for the inverse problem we assume that σ_a depends only on x.
A2. $\sigma_a \in L^\infty(M), k \in L^\infty(\{(y, v', v) \in M \times \Omega_y M \times \Omega_y M\})$, and $\|k\|_{L^\infty} \leq (2\pi \mathrm{diam}(M))^{-1}$.

Following the notation of Stefanov and Uhlmann (2003) we introduce the class

$$\mathcal{U}_{\Sigma,\varepsilon} = \{(\sigma_a(x), k(x, w', w)) : \|\sigma_a\|_{L^\infty} \leq \Sigma,\ \|k\|_{L^\infty} \leq \varepsilon, \text{ and } (\sigma_a, k) \text{ satisfies A1, A2}\}. \tag{2}$$

Theorem 1. *Let (M, g) be a two-dimensional Riemannian manifold with smooth, strictly convex boundary, satisfying assumptions* M1–M3: *(M, g) has curvature bounded above by κ_0, has no focal points, and in the case that $\kappa_0 > 0$ has diameter bounded by $\pi/(2\sqrt{\kappa_0})$.*

(1) *If (σ_a, k) satisfy* A1, A2 *then the metric g is uniquely determined by the associated albedo operator $\mathcal{A}$.*
(2) *Given $\Sigma > 0$ there exists $\varepsilon > 0$ such that any pair $(\sigma_a, k) \in \mathcal{U}_{\Sigma,\varepsilon}$ is uniquely determined, within $\mathcal{U}_{\Sigma,\varepsilon}$, by the associated albedo operator $\mathcal{A}$. Furthermore, ε can be chosen to be $\varepsilon = Ce^{-2A\Sigma}$, where C depends only on (M, g) and $A = \mathrm{diam}(M, g)$.*

As in Choulli and Stefanov (1996, 1999), Stefanov and Uhlmann (2003), and McDowall (2004), we find a singular expansion of $\alpha(x, v, x', v')$, the distributional kernel of the albedo operator $\mathcal{A}$. For dimensions $n \geq 3$ it was proven by McDowall (2004) (see also Choulli and Stefanov, 1999) that, writing $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, α_0 and α_1 are delta-type singularities supported on varieties of positive and differing codimensions in $\Gamma_+ \times \Gamma_-$. For dimension $n = 2$, however, McDowall (2004) did not show α_1 as distinguishable from the remainder, α_2. In the Euclidean case, the more precise analysis in Stefanov and Uhlmann (2003) of the singular decomposition allows one to show that the collision kernel k is uniquely determined by $\mathscr{A}$; we shall perform a similar analysis here for a domain with a Riemannian metric.

The determination of the metric g follows from the result of Pestov and Uhlmann (to appear) (see also Pestov and Uhlmann, 2004). They show that the scattering relation of (M, g) uniquely determines the metric g in dimension two when (M, g) is simple, up to a diffeomorphism that fixes the boundary of M. The scattering relation is the set of pairs $((x', v'), (x, v)) \in \Gamma_- \times \Gamma_+$, where (x, v) is the "exit" point and direction of the geodesic which "enters" M at x' with direction v'. In this work we simply observe that the leading term α_0 in the expansion of the kernel of $\mathscr{A}$ determines this scattering relation of (M, g).

This article is arranged as follows. In Section 2 we outline the construction of the solution to the forward problem and so define the albedo operator $\mathscr{A}$. We express $\mathscr{A}$ in terms of a Neumann series expansion for $(I + K)^{-1}$, where K is an integral operator, and we calculate explicitly the first two terms. In Section 3 we prove part one of Theorem 1 and then outline the method of proof of part two in order to motivate the estimates in Section 4. In Section 4 we obtain estimates for terms involving K^2 and K^3, which we use in Section 5 to complete the proof of Theorem 1.

2. The Forward Problem and the Albedo Operator

We define the following terms and notation. If $(x, v) \in \Omega M$ we denote by $\gamma_{(x,v)}(t)$ the geodesic that has $\gamma_{(x,v)}(0) = x$ and $\dot{\gamma}_{(x,v)}(0) = v$; we introduce the compressed notation

$$\vec{\gamma}_{(x,v)}(t) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)).$$

Define the "time to boundary" functions $\tau_\pm : \Omega M \to \mathbb{R}^+$ by

$$\tau_\pm(x, v) = \min\{t \geq 0 : \gamma_{(x,v)}(\pm t) \in \partial M\}$$

and set $\tau(x, v) = \tau_-(x, v) + \tau_+(x, v)$. Since (M, g) is simple, these functions are well-defined and finite. The operator $\mathscr{D}$ in (1) is the derivative along the geodesic flow and is defined by

$$\mathscr{D}f(x, v) = \frac{\partial}{\partial t}\bigg|_{t=0} f(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)). \tag{3}$$

If $(x^i, y^i)_{i=1}^n$ are local coordinates for ΩM with the (y^i) with respect to the natural basis $\left(\frac{\partial}{\partial x^i}\right)$, then in these coordinates

$$\mathscr{D}f = \frac{\partial f}{\partial x^i} y^i + \frac{\partial f}{\partial y^i}(-y^j y^k \Gamma^i_{jk}),$$

where Γ^i_{jk} are the Christoffel symbols of the Levi–Civita connection of g.

Definition 2. The *Liouville volume form* is the canonical volume form defined on ΩM, which is preserved under the geodesic flow of g. It is the product of the Riemannian volume $d\omega(x)$ on M and the surface element dv_x on $\Omega_x M$ induced from the Euclidean volume form in T_xM defined by the metric g at x (see, for example, Helgason, 1978; Katok and Hasselblatt, 1995).

Following the notation of McDowall (2004) we set

$$T_0 f = -\mathcal{D}f - \sigma_a f, \qquad T_1 f(x, v) = \int_{\Omega_x M} k(x, v', v) f(x, v')\,dv',$$

so that the transport Equation (1) reads $T_0 f + T_1 f = 0$. We begin by proving solvability of the forward problem and in the process obtain a series expansion for the albedo operator $\mathcal{A}$. In this section we do not need assumption A1. Since we seek the distribution kernel α of $\mathcal{A}$, it will suffice to study solvability of the boundary value problem for the transport equation

$$\begin{aligned} T_0 f + T_1 f &= 0 \qquad \text{in } \Omega M \\ f|_{\Gamma_-} &= f_- \end{aligned} \tag{4}$$

with smooth boundary functions $f_- \in C^\infty(\Gamma_-)$. We rewrite this as an integral equation.

In the absence of scattering, the homogeneous boundary value problem

$$\begin{aligned} T_0 f &= 0 \qquad \text{in } \Omega M \\ f|_{\Gamma_-} &= f_- \end{aligned} \tag{5}$$

has solution Lf_- where

$$Lf_-(x, v) = E(x, v, 0, -\tau_-(x, v)) f_-(\vec{\gamma}_{(x,v)}(-\tau_-(x, v))), \tag{6}$$

and here we introduce the notation

$$E(x, v, s, t) = \exp\left\{ \int_s^t \sigma_a(\vec{\gamma}_{(x,v)}(r))\,dr \right\}.$$

The exponent in $E(x, v, s, t)$ equals the total absorption along the geodesic $\gamma_{(x,v)}(r)$ between $r = s$ and $r = t$; note in particular that when $t < s$, the exponent is negative. Note further that $E(\cdot)$ is always a positive number.

Next, the inhomogeneous Dirichlet problem

$$\begin{aligned} T_0 f &= g \qquad \text{in } \Omega M \\ f|_{\Gamma_-} &= 0 \end{aligned} \tag{7}$$

has solution $T_0^{-1} g$, where

$$T_0^{-1} g(x, v) = -\int_0^{\tau_-(x,v)} E(x, v, 0, t - \tau_-(x, v))\, g(\vec{\gamma}_{(x,v)}(t - \tau_-(x, v)))\,dt. \tag{8}$$

To see this, one need only notice that

$$\begin{aligned} \mathcal{D}(E(x, v, 0, -\tau_-(x, v)) f(x, v)) &= -E(x, v, 0, -\tau_-(x, v))(\mathcal{D}f(x, v) + \sigma_a(x, v) f(x, v)) \\ &= -E(x, v, 0, -\tau_-(x, v)) T_0 f. \end{aligned}$$

Thus, if we define the operator K by

$$Kf(x, v) = -\int_0^{\tau_-(x,v)} E(x, v, 0, t - \tau_-(x, v))(T_1 f)(\vec{\gamma}_{(x,v)}(t - \tau_-(x, v)))dt,$$

we have that $K = T_0^{-1}T_1$ and (4) is equivalent to

$$(I + K)f = Lf_-.$$

Now

$$\|Kf\|_{L^\infty(\Omega M)} \leq 2\pi \operatorname{diam}(M)\|k\|_{L^\infty(\Omega M)}\|f\|_{L^\infty(\Omega M)}$$

and so if $2\pi\mathrm{diam}(M)\|k\|_{L^\infty} < 1$, then $I + K$ is invertible on $L^\infty(\Omega M)$. If $f_- \in C^\infty(\Gamma_-)$ then Lf_- clearly has a well-defined trace $Lf_-|_{(x,v)\in\Gamma_+}$; repeated application of K preserves this and so by considering the series representation of $(I + K)^{-1}$ we see that $f = (I + K)^{-1}Lf_-$ has a well-defined trace and that $\mathscr{A}$ maps $C^\infty(\Gamma_-)$ into $L^\infty(\Gamma_+)$.

Write the solution f as

$$f = Lf_- - KLf_- + (I + K)^{-1}K^2Lf_-. \tag{9}$$

We wish to express f as

$$f(x, v) = \int_{\Gamma_-} (\phi_0 + \phi_1 + \phi_2)(x, v, x', v')f_-(x', v')d\mu(x', v'). \tag{10}$$

That is to say, we will express f in terms of an expansion of the fundamental solution ϕ to the transport equation with a delta boundary condition

$$\begin{aligned} T_0\phi + T_1\phi = 0 \qquad &\text{in } \Omega M \\ \phi|_{\Gamma_-} = \phi_- \end{aligned} \tag{11}$$

with

$$\phi_-(x', v'; x'', v'') = \frac{1}{|\langle n_{x'}, v'\rangle|}\delta_{(x'',v'')}(x', v').$$

Here, $\delta_{(x'',v'')}(x', v')$ is the Dirac delta distribution with respect to the measure $d\mu(x', v')$. The measure $d\mu(x', v')$ is obtained via pulling back the Liouville volume form of Definition 2 to $\Gamma_\pm$ by the geodesic flow. See McDowall (2004) for a more explicit description. Formally, we have

$$\phi_0 = L\phi_-, \qquad \phi_1 = -KL\phi_-, \qquad \text{and} \qquad \phi_2 = (I + K)^{-1}K^2L\phi_- \tag{12}$$

and the kernel α of $\mathscr{A}$ is given by $\alpha = \phi|_{(x,v)\in\Gamma_+}$, and $\alpha_j = \phi_j|_{(x,v)\in\Gamma_+}$, $j = 1, 2, 3$.

It was demonstrated in Proposition 3.1 of McDowall (2004) that

$$Lf_-(x, v) = \int_{\Gamma_-} \phi_0(x, v, x', v')f_-(x', v')d\mu(x', v'), \tag{13}$$

with

$$\phi_0(x, v, x', v') = \int_0^{\tau_+(x',v')} E(x, v, 0, -\tau_-(x, v))\delta_{(x,v)}(\vec{\gamma}_{(x',v')}(t))dt$$

where $\delta_{(x,v)}(y, w)$ is the delta distribution with respect to the Liouville measure $dw_y\, d\omega(y)$ on ΩM. From this we have $\alpha_0 = \phi_0|_{(x,v)\in\Gamma_+}$.

In Proposition 4 below, we shall perform a change of variables of integration defined in terms of the geodesic flow of g. Lemma 3 facilitates the computation of the Jacobian of this change of variables. Let $(x', v') \in \Omega M$ be fixed, and let $s_0 \geq 0$ be such that $y = \gamma_{(x',v')}(s_0) \in M$. Define $v'^{\perp}$ so that $\{v'^{\perp}, v'\}$ is a positive orthonormal basis for $T_{x'}M$. Define the geodesic flow $\varphi_{s_0} : \Omega M \to \Omega M$, $\varphi_{s_0}(\hat{x}, \hat{v}) = \vec{\gamma}_{(\hat{x},\hat{v})}(s_0)$ defined for $(\hat{x}, \hat{x})$ near (x', v'). In the lemma below, D_t denotes covariant differentiation along the curve $\gamma_{(x',v')}$ and $R(U, V)W = \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W$ is the curvature tensor with respect to the Levi–Civita connection ∇ of g.

Lemma 3. *Let J_{ξ_H} be a vector field along $\gamma_{(x',v')}$ satisfying the Jacobi equation*

$$D_t^2 J_{\xi_H} + R(J_{\xi_H}, \dot{\gamma}_{(x',v')})\dot{\gamma}_{(x',v')} = 0$$

$$J_{\xi_H}(0) = v'^{\perp}, \qquad D_t J_{\xi_H}(0) = 0$$

and let J_{ξ_V} be the vector field along $\gamma_{(x',v')}$ satisfying the same Jacobi equation but with initial conditions

$$J_{\xi_V}(0) = 0, \qquad D_t J_{\xi_V}(0) = v'^{\perp}.$$

Then

$$(d\varphi_{s_0})_{(x',v')}(\xi_H) = (J_{\xi_H}(s_0), \dot{J}_{\xi_H}(s_0))$$

$$(d\varphi_{s_0})_{(x',v')}(\xi_V) = (J_{\xi_V}(s_0), \dot{J}_{\xi_V}(s_0)).$$

Proof. For general $\xi \in T_{(x',v')}\Omega M$ it is proven in Paternain (1999) that $(d\varphi_{s_0})_{(x',v')}(\xi)$ can be described in terms of Jacobi fields whose initial conditions are given in terms of the horizontal and vertical projections of ξ. The claim of the lemma is this result applied to the vectors ξ_H and ξ_V defined as follows.

The vector $\xi_H \in T_{(x',v')}\Omega_y M$ belongs to the so-called *horizontal* subbundle: Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve adapted to $v'^{\perp} \in \Omega_{x'}M$, i.e., $\dot{\alpha}(0) = v'^{\perp}$. Let $Z(r)$ be the parallel transport of v' along α, and define $\beta : (-\varepsilon, \varepsilon) \to \Omega M$ by $\beta(r) = (\alpha(r), Z(r))$. Then $\xi_H = \dot{\beta}(0)$.

Similarly, we define $\xi_V \in T_{(x',v')}\Omega_{x'}M$ to be the vector in the *vertical* subbundle given as the tangent vector to the curve $\beta : (-\varepsilon, \varepsilon) \to TM$, $\beta(r) = (x', v' + rv'^{\perp})$. □

From this point on in the exposition we will frequently not present the exact point in $\Omega M \times \mathbb{R}^2$ where E is evaluated, choosing instead to simply write "$E(\cdot)$." We will not need the omitted information.

Proposition 4. *The second term in* (9) *is given by*

$$-KLf_-(x, v) = \int_{\Gamma_-} \phi_1(x, v, x', v')f_-(x', v')d\mu(x', v') \tag{14}$$

with

$$\begin{aligned}\phi_1(x, v, x', v') = {}& \chi(x, v; x', v')E(\cdot)E(\cdot)\mathscr{J}(x, v; x', v')\\ &\times \frac{k(\vec{\gamma}_{(x',v')}(s(x', v')), \dot{\gamma}_{(x,v)}(t(x', v') - \tau_-(x, v)))}{|\sin(\psi(x, v; x', v'))|}\end{aligned}$$

where $\chi(x, v; x', v') = 1$ *if the geodesics* $\gamma_{(x',v')}(s)$ *and* $\gamma_{(x,v)}(t - \tau_-(x, v))$ *intersect for* $s = s(x', v') > 0$ *and* $t = t(x, v) > 0$, *and* $\chi(x, v; x', v') = 0$ *otherwise, and where* $\psi(x, v; x', v')$ *is the angle between the tangent vectors at the point of intersection. The function* $\mathscr{J}$ *is uniformly bounded* $0 < m_1 \leq \mathscr{J} \leq m_2$.

Thus the second term in the expansion for the distribution kernel α of $\mathscr{A}$ is $\alpha_1 = \phi_1|_{(x,v)\in\Gamma_+}$. Recall that ϕ_0 involves a delta-type singularity; as is evident from the proof which follows, the singularity in ϕ_1 in integrable, and thus ϕ_1 is distinguishable from ϕ_0.

Proof. By definition,

$$\begin{aligned}-KLf_-(x, v) = \int_0^{\tau_-(x,v)} E(\cdot) \int_{\Omega_y M} & k(y, w, \dot{\gamma}_{(x,v)}(t - \tau_-(x, v)))\\ &\times E(\cdot)f_-(\vec{\gamma}_{(y,w)}(-\tau_-(y, w)))dw\, dt\end{aligned}$$

where $y = y(t) = \gamma_{(x,v)}(t - \tau_-(x, v))$. We wish to rewrite this integral in terms of an integral with respect to the variables $x' \in \partial M$ and $v' \in \Omega_{x'}M$. Let us define a family of indicator functions $\chi : \Omega M \to \{0, 1\}$, parameterized by (x, v), by

$$\chi(x, v; x', v') = \begin{cases} 1 & \text{if } \gamma_{(x',v')}(s) = \gamma_{(x,v)}(t - \tau_-(x, v)) = y(t)\\ & \text{for some unique } s > 0, t > 0,\\ 0 & \text{otherwise.}\end{cases}$$

On the support of χ we have well-defined functions $s(x', v')$, $t(x', v')$ defined to be the s and t, respectively, which appear in the definition of χ. Then, also on the support of χ, the change of variables

$$(t, w) = \Phi(x', v') = (t(x', v'), \dot{\gamma}_{(x',v')}(s(x', v')))$$

is well-defined and smooth. We now seek to compute the Jacobian of this change of variables.

We first specialize to the case where ∂M is perpendicular to $\gamma_{(x',v')}$ at x' and $\gamma_{(x,v)}$ is perpendicular to $\gamma_{(x',v')}$ at $y(t)$. For a fixed $(x', v') \in \operatorname{supp}\chi$, $x' \in \partial M$, let $(y, w) = \vec{\gamma}_{(x',v')}(s(x', v'))$ and let $w^\perp \in \Omega_y M$ be such that $\{w^\perp, w\}$ is a positive orthonormal basis for T_yM. Let $J_{\xi_H} = J^{(x',v')}_{\xi_H}$ and $J_{\xi_V} = J^{(x',v')}_{\xi_V}$ be the Jacobi fields along $\gamma_{(x',v')}$ given in Lemma 3 above, where we are taking $s_0 = s(x', v')$. Extend the metric g to an ε neighborhood of ∂M near x'. Now let $(x_1', x_2', r', \theta')$ be local coordinates near $(x', v') = (0, 0, 1, \pi/2)$ so that $(x', v'^\perp) = (0, 0, 1, 0)$, so that $\gamma_{(x',v'^\perp)} = \{x_2' = 0\}$, and so that $(x_1', 0, 1, \pi/2) = (\gamma_{(x',v'^\perp)}(x_1'), \mathscr{P}(v'; x', \gamma_{(x',v'^\perp)}(x_1')))$. Here $\mathscr{P}(v'; x', \gamma_{(x',v'^\perp)}(x_1'))$ is simply the parallel transport of v' along $\gamma_{(x',v'^\perp)}$ from x' to $\gamma_{(x',v'^\perp)}(x_1')$. Thus the (x_1', x_2') coordinates are "geodesic normal coordinates" along $\gamma_{(x',v'^\perp)}$. Similarly, let (y_1, y_2, r, θ) be coordinates local to (y, w), so that $(y, w^{\perp}) = (0, 0, 1, 0)$ and $\gamma_{(y,w^{\perp})} = \{y_2 = 0\}$. In these coordinates,

$$J_{\xi_H}(s_0) = \frac{\partial y_1}{\partial x_1'}\frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_1'}\frac{\partial}{\partial y_2} = \frac{\partial y_1}{\partial x_1'}\frac{\partial}{\partial y_1},$$

since J_{ξ_H} is perpendicular along $\gamma_{(x',v')}$ and so $\frac{\partial y_2}{\partial x_1'} = 0$ at y. Next,

$$\dot{J}_{\xi_H}(s_0) = \frac{\partial r}{\partial x_1'}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x_1'}\frac{\partial}{\partial \theta} = \frac{\partial \theta}{\partial x_1'}\frac{\partial}{\partial \theta},$$

since J_{ξ_H} is a Jacobi field from a variation of geodesics, all of which are parameterized at unit speed. Similarly, we have

$$J_{\xi_V}(s_0) = \frac{\partial y_1}{\partial \theta'}\frac{\partial}{\partial y_1}, \qquad \text{and} \qquad \dot{J}_{\xi_V}(s_0) = \frac{\partial \theta}{\partial \theta'}\frac{\partial}{\partial \theta}.$$

Let us define $\widetilde{\Phi} : \mathbb{R} \times S^1 \to \mathbb{R}^3 \times S^1$, $(x_1', \theta') \mapsto (y_1, y_2, r, \theta)$ equal to $(\vec{\gamma}_{(x_1',\theta')}(s_0))$ in the (y_1, y_2, r, θ) local coordinates where we are abusing notation in the use of $\vec{\gamma}_{(x_1',\theta')}(s_0)$, but where we trust that the meaning is clear to the reader. Then the differential of this map at $(0, \pi/2)$ is given by

$$(d\widetilde{\Phi})_{(0,\pi/2)} = \begin{pmatrix} \dfrac{\partial y_1}{\partial x_1'} & \dfrac{\partial \theta}{\partial x_1'} \\[2ex] \dfrac{\partial y_1}{\partial \theta'} & \dfrac{\partial \theta}{\partial \theta'} \end{pmatrix}.$$

Notice that $\det(d\widetilde{\Phi})_{(0,\pi/2)}$ cannot be zero, for if it were we would have that for some $\lambda \in \mathbb{R}$, $(J_{\xi_H}(s_0), \dot{J}_{\xi_H}(s_0)) = \lambda(J_{\xi_V}(s_0), \dot{J}_{\xi_V}(s_0))$, and then $J_{\xi_H} \equiv J_{\xi_V}$, a contradiction.

For the general case let $\psi(x', v') = \cos^{-1}\langle \dot{\gamma}_{(x,v)}(t - \tau_-(x, v)), \gamma_{(x',v')}(s(x', v'))\rangle$ be the angle with respect to the metric g between the geodesics $\gamma_{(x,v)}$ and $\gamma_{(x',v')}$ intersecting at $y = y(t(x', v'))$. If J_Φ denotes the Jacobian of the change of variables Φ then we have in local coordinates

$$J_\Phi(x', v') = \begin{pmatrix} \dfrac{|\langle n_{x'}, v'\rangle_g|}{\sin(\psi(x', v'))}\dfrac{\partial y_1}{\partial x_1'} & \dfrac{1}{\sin(\psi(x', v'))}\dfrac{\partial \theta}{\partial x_1'} \\[3ex] |\langle n_{x'}, v'\rangle_g|\dfrac{\partial y_1}{\partial \theta'} & \dfrac{\partial \theta}{\partial \theta'} \end{pmatrix}$$

where $n_{x'}$ is the outward pointing unit normal vector to ∂M at $x' \in \partial M$. Thus

$$|\det J_\Phi| = \frac{|\langle n_{x'}, v'\rangle_g|}{|\sin(\psi(x', v'))|}\left|\frac{\partial y_1}{\partial x_1'}\frac{\partial \theta}{\partial \theta'} - \frac{\partial y_1}{\partial \theta'}\frac{\partial \theta}{\partial x_1'}\right| = \frac{|\langle n_{x'}, v'\rangle_g|}{|\sin(\psi(x', v'))|}\mathcal{J}(x', v'),$$

say. From the remark above, we have $\mathcal{J}(x', v') \neq 0$.

The above calculation was performed with (x, v) fixed. In fact every term in J_Φ is dependent on, or parameterized by, (x, v), and we introduce the notation $\mathcal{J}(x, v; x', v')$ and $\psi(x, v; x', v')$, where the order in which the variables have been written is chosen to be consistent with the notation elsewhere in this article. Note that the second of these terms is the angle with respect to g between the geodesics $\gamma_{(x',v')}(s)$ and $\gamma_{(x,v)}(t-\tau_-(x,v))$ at the point of intersection, should it exist for $s>0$ and $t>0$. It is not difficult to see that from the compactness of $\Omega\overline{M}$ that there are constants $0<m_1<m_2$ such that $m_1 \le \mathcal{J}(x,v;x',v') \le m_2$ uniformly. Note that this does not rely on the fact that the metric is simple; it relies only on the fact that geodesics are uniquely determined by an element of ΩM.

If we denote $\Omega_{x'}^- = \{v' \in \Omega_x M : \langle n_{x'}, v'\rangle < 0\}$ and let $d\sigma(x')$ be the induced Riemannian measure on ∂M, we now have

$$\begin{aligned}
-KLf_-(x,v) &= \int_{\partial M}\int_{\Omega_{x'}^-} \chi(x,v;x',v')E(\cdot)E(\cdot) \\
&\quad \times k(\vec{\gamma}_{(x',v')}(s(x',v')), \dot{\gamma}_{(x,v)}(t(x',v')-\tau_-(x,v)))f_-(x',v') \\
&\quad \times \frac{\mathcal{J}(x,v;x',v')}{|\sin(\psi(x,v;x',v'))|}|\langle n_{x'},v'\rangle_g|dv'\,d\sigma(x') \\
&= \int_{\Gamma_-} \phi_1(x,v,x',v')f_-(x',v')d\mu(x',v')
\end{aligned}$$

where $\phi_1(x,v,x',v')$ is as in the statement of the proposition and where we have used that $|\langle n_{x'},v'\rangle_g|dv'\,d\sigma(x') = d\mu(x',v')$. $\square$

3. Outline of Proof of Theorem 1

Suppose that we have two albedo operators $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ coming from measurements made on manifolds (M,g) and $(M,\tilde{g})$ with material parameters (σ_a,k) and $(\tilde{\sigma}_a,\tilde{k})$, respectively. Suppose that $\mathcal{A}=\widetilde{\mathcal{A}}$. Let all the operators and functions $T_0, T_1, J, E, K, \alpha, \alpha_j$ defined thus far in terms of (σ_a,k) have their counterparts for $(\tilde{\sigma}_a,\tilde{k})$ denoted by $\widetilde{T}_0, \widetilde{T}_1, \widetilde{J}, \widetilde{E}, \widetilde{K}, \tilde{\alpha}, \tilde{\alpha}_j$. With $\alpha=\tilde{\alpha}$ and assuming A1, from McDowall (2004) we have that $\alpha_0=\alpha_0'$; in fact

$$\alpha_0(x,v,x',v') = E(x,v,-\tau_-(x,v),0)\delta_{\{\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\}}(x',v')$$

(and similarly for $\tilde{\alpha}_0$), and so we see that α_0 determines the scattering relation

$$\mathcal{S} = \{((x',v'),(\gamma_{(x',v')}(\tau_+(x',v')),\dot{\gamma}_{(x',v')}(\tau_+(x',v'))))\}$$

for (M,g). Thus $\alpha_0=\tilde{\alpha}_0$ implies that $\mathcal{S}=\widetilde{\mathcal{S}}$, where $\widetilde{\mathcal{S}}$ is the relation for $(M,\tilde{g})$. In Pestov and Uhlmann (to appear) (see also Pestov and Uhlmann, 2004) it is proven that this implies that $g=\tilde{g}$, of course up to a choice of coordinates, that is, a diffeomorphism $\varphi:\overline{M}\to\overline{M}$ with $\varphi|_{\partial M}=\mathrm{Id}|_{\partial M}$. This completes the proof of part one of Theorem 1.

To motivate the analysis that follows, and with Proposition 4 at our disposal, we outline the approach we shall take to prove part two of Theorem 1. With $g=\tilde{g}$ it follows from McDowall (2004) that $\sigma_a=\tilde{\sigma}_a$ and so $(T_0,J,E)=(\widetilde{T}_0,\widetilde{J},\widetilde{E})$.

Combining this with Proposition 4 we then have

$$\begin{aligned}(\alpha_2 - \tilde{\alpha}_2)(x, v, x', v') &= (\tilde{\alpha}_1 - \alpha_1)(x, v, x', v') \\ &= \chi(x, v; x', v')E(\cdot)E(\cdot)\mathcal{J}(\cdot) \\ &\quad \times \frac{(\tilde{k} - k)(\vec{\gamma}_{(x',v')}(s(x', v')), \dot{\gamma}_{(x,v)}(t(x', v') - \tau_-(x, v)))}{|\sin(\psi(x, v; x', v'))|}\end{aligned} \tag{15}$$

for $(x, v, x', v') \in \Gamma_+ \times \Gamma_-$. We refer the reader to the statement of Proposition 4 for the definitions of χ, s, t, and ψ. Recall that $\mathcal{J}$ is a function uniformly bounded on $\Gamma_+ \times \Gamma_-$. In what follows we will obtain another estimate for $\alpha_2 - \tilde{\alpha}_2$ of the form

$$\|\sin\psi\,(\alpha_2 - \tilde{\alpha}_2)\|_{L^\infty(\Gamma_+\times\Gamma_-)} \leq C\varepsilon\|k - \tilde{k}\|_{L^\infty}, \tag{16}$$

with $C > 0$ a constant depending only on the geometry of the manifold (M, g). Combining (15) and (16), we will obtain

$$\|k - \tilde{k}\|_{L^\infty} \leq C\varepsilon\|k - \tilde{k}\|_{L^\infty},$$

which implies that $k = \tilde{k}$ if ε is sufficiently small. The principal observation to make here is that in relating $(\alpha_2 - \tilde{\alpha}_2)(x, v, x', v')$ to $k - \tilde{k}$, we need only consider $(x, v, x', v') \in \Gamma_+ \times \Gamma_-$ such that the geodesics $\gamma_{(x,v)}$ and $\gamma_{(x',v')}$ intersect. We see this due to the presence of the indicator function χ, and due to the fact that to identify $k(y, w, w')$ at $(y, w, w') \in M \times \Omega_y M \times \Omega_y M$ we should follow the geodesics $\gamma_{(y,w)}$ and $\gamma_{(y,w')}$ backward and forward, respectively, to the boundary.

Now, following the decomposition in Stefanov and Uhlmann (2003), by (12) and the statement immediately following (12) we write

$$\begin{aligned}\alpha_2 - \tilde{\alpha}_2 &= (\phi_2 - \tilde{\phi}_2)|_{(x,v)\in\Gamma_+} \\ &= ((I + K)^{-1}K^2\phi_0 - (I + \widetilde{K})^{-1}\widetilde{K}^2\phi_0)|_{(x,v)\in\Gamma_+} \\ &= ((I + K)^{-1}(K^2 - \widetilde{K}^2)\phi_0 + (I + \widetilde{K})^{-1}(\widetilde{K} - K)(I + K)^{-1}\widetilde{K}^2\phi_0)|_{(x,v)\in\Gamma_+}.\end{aligned} \tag{17}$$

Writing $K^2 - \widetilde{K}^2 = K(K - \widetilde{K}) + (K - \widetilde{K})\widetilde{K}$ we see that we must estimate $K(K - \widetilde{K})\phi_0$, which is the content of Proposition 7 (and Lemma 9) below for (x, v, x', v') such that the corresponding geodesics intersect as mentioned above. We estimate the final term in Proposition 8 (and Lemma 10).

4. Estimates for $K^2\phi_0$ and $K^3\phi_0$

With a view toward Proposition 7 below, we introduce the following notation. Let $(x_0', v_0') \in \Gamma_-$ be fixed; this is the location of the delta source in (11). For a given $0 \leq t \leq \tau_+(x_0', v_0')$, let $\gamma(s) = \gamma^t(s)$ be the unit speed geodesic with $\gamma(0) = \gamma_{(x_0',v_0')}(t)$ joining $\gamma_{(x_0',v_0')}(t)$ to x. Let $d(\gamma_{(x_0',v_0')}(t), x)$ be the geodesic distance between these two points, so $\gamma(d(\gamma_{(x_0',v_0')}(t), x)) = x$. Define

$$\bar{v} = \bar{v}(t, x) = \dot{\gamma}(0), \qquad \text{and} \qquad \hat{v} = \hat{v}(t, x) = \dot{\gamma}(d(\gamma_{(x_0',v_0')}(t), x)). \tag{18}$$

Next, let $X = X_{(t,x)}$ be the Jacobi field along $\gamma(d(\gamma_{(x_0',v_0')}(t), x) - s)$ $(0 \le s \le d(\gamma_{(x_0',v_0')}(t), x))$ satisfying $X(0) = 0$ and $\dot{X}(0) = \hat{v}^{\perp}$, where $\{\hat{v}^{\perp}, \hat{v}\}$ form a positive orthonormal basis for T_xM. Then we have

Proposition 5. *For $(x, v) \in \Omega M$ and with the notation defined above,*

$$\begin{aligned}\widetilde{T}_1 K\phi_0(x, v) &= \widetilde{T}_1 KL\phi_-(x, v)\\ &= \int_0^{\tau_+(x_0',v_0')} E(\cdot)E(\cdot)\tilde{k}(x, \hat{v}, v)k(\vec{\gamma}_{(x_0',v_0')}(t), \bar{v})\frac{1}{|X_{\hat{v}}(d(x, \gamma_{(x_0',v_0')}(t)))|}\,dt.\end{aligned}$$

Proof. For an arbitrary $f_-(x', v')$ we have

$$\begin{aligned}\widetilde{T}_1 KLf_-(x, v) = \int_{\Omega_x M}\int_0^{\tau_-(x,v'')}\int_{\Omega_{y(t,v'')}M} & E(\cdot)\tilde{k}(x, v'', v)k(y(t, v''), \tilde{v}, \dot{y}(t, v''))\\ &\times Jf_-(y(t, v''), \tilde{v})\,d\tilde{v}\,dt\,dv'',\end{aligned}$$

where $y(t, v'') = \gamma_{(x,v'')}(t - \tau_-(x, v''))$. We now make the change of variables $(t, v'') \mapsto y(t, v'')$. The change of volume element is computed to be $dy = |Y_{(x,y)}(d(x, y))|\,dt\,dv''$, where $Y = Y_{(x,y)}$ is the Jacobi field along $\gamma_{(x,-v'')}(r)$, $0 \le r \le d(x, y)$ with $Y(0) = 0$ and $\dot{Y}(0) = v''^{\perp}$, where $\{v''^{\perp}, v''\}$ is a positive orthonormal basis for T_xM. Thus

$$\begin{aligned}\widetilde{T}_1 KLf_-(x, v) = \int_M\int_{\Omega_y M} & E(\cdot)\tilde{k}(x, w(x, y), v)k(y, \tilde{v}, \tilde{w}(x, y))\\ &\times Jf_-(y, \tilde{v})\frac{1}{|Y(d(x, y))|}\,d\tilde{v}\,d\omega(y)\end{aligned}$$

where $w(x, y)$ and $\tilde{w}(x, y)$ are the unit tangent vectors at y and x, respectively, of the geodesic joining y to x. Note that the singularity at $x = y$ in the integral above is of the type $1/r$ in two dimensions and so is integrable. As in Lemma 2.2 of McDowall (2004), and changing variables $(y, \tilde{v}) \mapsto \vec{\gamma}_{(x',v')}(t)$, we have

$$\begin{aligned}\widetilde{T}_1 KLf_-(x, v) = \int_{\Gamma_-}\int_0^{\tau_+(x',v')} & E(\cdot)\tilde{k}(x, w(x, \gamma_{(x',v')}(t)), v)k(\vec{\gamma}_{(x',v')}(t), \tilde{w}(x, \gamma_{(x',v')}(t)))\\ &\times \frac{1}{|Y(d(x, \gamma_{(x',v')}(t)))|}Jf_-(\vec{\gamma}_{(x',v')}(t))\,dt\,d\mu(x', v').\end{aligned}$$

Finally, using $dt\,d\mu(x', v') = |\langle n_{x'}, v'\rangle|\,d\sigma(x')dv'$ and setting $f_- = \phi_-$, we obtain

$$\begin{aligned}\widetilde{T}_1 KL\phi_-(x, v) &= \int_{\partial M}\int_{\Omega_{x'}^- M}\int_0^{\tau_+(x',v')} E(\cdot)E(\cdot)\tilde{k}(x, w(x, \gamma_{(x',v')}(t)), v)\\ &\qquad\times k(\vec{\gamma}_{(x',v')}(t), \tilde{w}(x, \gamma_{(x',v')}(t)))\delta_{x_0'}(x')\delta_{v_0'}(v')\\ &\qquad\times \frac{1}{|Y(d(x, \gamma_{(x',v')}(t)))|}\,dt\,dv'\,d\sigma(x')\\ &= \int_0^{\tau_+(x_0',v_0')} E(\cdot)E(\cdot)\tilde{k}(x, \hat{v}, v)k(\vec{\gamma}_{(x_0',v_0')}(t), \bar{v})\frac{1}{|X_{\hat{v}}(d(x, \gamma_{(x_0',v_0')}(t)))|}\,dt,\end{aligned}$$

where $\hat{v}$, $\bar{v}$, and $X_{\hat{v}}$ are as defined above the statement of the proposition. □

In the estimates proven below we will repeatedly make use of the fact that $E(\cdot) \leq 1$.

Lemma 6. *Assuming* M1–M3 *and* A1, A2, *for almost every* $(x, v, x_0', v_0') \in \Omega M \times \Gamma_-$ *and for almost every* $(x, v, x_0', v_0') \in \Gamma_+ \times \Gamma_-$,

$$|\widetilde{T}_1 KL\phi_-(x, v, x_0', v_0')| \leq \begin{cases} \dfrac{\pi}{\cos(\sqrt{\kappa_0}A)} \|\tilde{k}\|_{L^\infty} \|k\|_{L^\infty} \left(C_+ + \log \dfrac{A}{\nu} \right) & \kappa_0 > 0, \\ 2\|\tilde{k}\|_{L^\infty} \|k\|_{L^\infty} \left(1 + \log \dfrac{A}{\nu} \right) & \kappa_0 = 0, \\ \|\tilde{k}\|_{L^\infty} \|k\|_{L^\infty} \left(C_- + \log \dfrac{A}{\nu} \right) & \kappa_0 < 0 \end{cases} \tag{19}$$

where ν *is the minimal distance of* x *from the geodesic* $\gamma_{(x_0', v_0')}(t)$, $0 \leq t \leq \tau_+(x_0', v_0')$, *and where* $C_+ = \log(3/\cos(\sqrt{\kappa_0}A))$, $C_- = \log(3/\cosh(\sqrt{-\kappa_0}A))$. *Recall that* A *is the diameter of* M.

Proof. The Rauch comparison lemma gives

$$|X_{\hat{v}}(d(x, \gamma_{(x_0', v_0')}(t)))| \geq \begin{cases} \dfrac{1}{\sqrt{\kappa_0}} \sin\left(\sqrt{\kappa_0}\, d(x, \gamma_{(x_0', v_0')}(t)) \right) & \kappa_0 > 0, \\ d(x, \gamma_{(x_0', v_0')}(t)) & \kappa_0 = 0, \\ \dfrac{1}{\sqrt{-\kappa_0}} \sinh\left(\sqrt{-\kappa_0}\, d(x, \gamma_{(x_0', v_0')}(t)) \right) & \kappa_0 < 0. \end{cases} \tag{20}$$

Let us abbreviate $\gamma_{(x_0', v_0')}(\cdot)$ by $\gamma_0(\cdot)$ for the moment. Let $0 \leq t^* \leq \tau_+(x_0', v_0')$ be such that $\gamma_0(t^*)$ minimizes the geodesic distance from $\gamma_0(t)$ to x, and let $\nu = d(x, \gamma_0(t^*))$. Consider the geodesic triangle with vertices x, $\gamma_0(t^*)$, and $\gamma_0(t)$, and let α^* be the (interior) angle between the geodesic $\gamma_0(\cdot)$ and the geodesic joining $\gamma_0(t^*)$ and x. We compare this to the triangle on the manifold of constant curvature κ_0, which shares the same angle α^* and adjacent side lengths ν and $|t - t^*|$. Then $d(x, \gamma_0(t))$ is at least as great as the length of the third side of the comparison triangle, d_{κ_0} say.

Consider the case when $\kappa_0 > 0$. The law of cosines on the sphere of radius $1/\sqrt{\kappa_0}$ reads

$$\begin{aligned} \cos(\sqrt{\kappa_0}\, d_{\kappa_0}) &= \cos(\sqrt{\kappa_0}|t - t^*|) \cos(\sqrt{\kappa_0}\, \nu) + \cos(\alpha^*) \sin(\sqrt{\kappa_0}|t - t^*|) \sin(\sqrt{\kappa_0}\, \nu) \\ &\leq \cos(\sqrt{\kappa_0}|t - t^*|) \cos(\sqrt{\kappa_0}\, \nu), \end{aligned}$$

where the inequality follows from $\alpha^* \geq \pi/2$; this is easily seen by considering the possible location of x relative to the endpoints of γ_0. From this we obtain

$$\begin{aligned} \sin(\sqrt{\kappa_0}\, d(x, \gamma_0(t))) &= \sqrt{1 - \cos^2(\sqrt{\kappa_0}\, d(x, \gamma_0(t)))} \\ &\geq \sqrt{1 - \cos^2(\sqrt{\kappa_0}|t - t^*|) \cos^2(\sqrt{\kappa_0}\, \nu)} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\sin^2(\sqrt{\kappa_0}\, v) + \sin^2(\sqrt{\kappa_0}|t - t^*|)\cos^2(\sqrt{\kappa_0}\, v)} \\
&\geq \frac{2\sqrt{\kappa_0}}{\pi}\sqrt{v^2 + |t - t^*|^2\cos^2(\sqrt{\kappa_0}A)},
\end{aligned}
$$

since $v < A$. Combining this with (20) we thus obtain

$$
\begin{aligned}
&\int_0^{\tau_+(x_0', v_0')} \frac{1}{|X_{\hat{v}}(d(x, \gamma_{(x_0', v_0')}(t)))|}\, dt \\
&\leq \frac{\pi}{2}\int_{-t^*}^{A-t^*} \frac{1}{\sqrt{v^2 + t^2\cos^2(\sqrt{\kappa_0}A)}}\, dt \\
&\leq \pi\int_0^{A} \frac{1}{\sqrt{v^2 + t^2\cos^2(\sqrt{\kappa_0}A)}}\, dt \\
&= \frac{\pi}{\cos(\sqrt{\kappa_0}A)}\log\left(\frac{A\cos(\sqrt{\kappa_0}A)}{v} + \sqrt{\frac{A^2\cos^2(\sqrt{\kappa_0}A)}{v^2} + 1}\right) \\
&\leq \frac{\pi}{\cos(\sqrt{\kappa_0}A)}\left(C_+ + \log\frac{A}{v}\right),
\end{aligned}
$$

where $C_+ = \log(3/\cos(\sqrt{\kappa_0}A)$. Note that the assumption on the diameter $A < \pi/(2\sqrt{\kappa_0})$ is used here. The last estimate is obtained as follows: let $\alpha = \cos(\sqrt{\kappa_0}A)$. Since $\alpha \leq 1$ and $A/v \geq 1$, one easily obtains

$$
\left(\frac{3}{\alpha^2} - 1\right)^2 \geq \frac{1}{\alpha^2} + 1 \implies \frac{A^2\alpha^2}{v^2} \geq \frac{1}{\left(\frac{3}{\alpha^2} - 1\right)^2 - 1}
$$

and so

$$
\frac{A\alpha}{v}\left(\frac{3}{\alpha^2} - 1\right) \geq \sqrt{\frac{A^2\alpha^2}{v^2} + 1} \implies \frac{A\alpha}{v} + \sqrt{\frac{A^2\alpha^2}{v^2} + 1} \leq \frac{3}{\alpha}\frac{A}{v}.
$$

Combining this estimate with Proposition 5 we obtain (19) for $\kappa_0 > 0$.

When $\kappa_0 < 0$ we use the law of cosines on the hyperbolic plane of constant curvature κ_0 as above to obtain

$$
\sinh(\sqrt{\kappa_0}\, d(x, \gamma_0(t))) \geq \sqrt{-\kappa_0}\sqrt{v^2 + |t - t^*|^2\cosh^2(\sqrt{-\kappa_0}\, v)}
$$

and then

$$
\int_0^{\tau_+(x_0', v_0')} \frac{1}{|X_{\hat{v}}(d(x, \gamma_{(x_0', v_0')}(t)))|}\, dt \leq \frac{1}{\cosh(\sqrt{-\kappa_0}\, v)}\left(C_- + \log\frac{A}{v}\right)
$$

with $C_- = \log(3/\cosh(\sqrt{-\kappa_0}A))$. This yields (19) for $\kappa_0 < 0$.

When $\kappa_0 = 0$, we have $d(x, \gamma_0(t)) = \sqrt{v^2 + |t - t^*|^2}$ and

$$
\int_0^{\tau_+(x_0', v_0')} \frac{1}{|X_{\hat{v}}(d(x, \gamma_{(x_0', v_0')}(t)))|}\, dt \leq 2\left(1 + \log\frac{A}{v}\right).
$$

Notice that no restriction on A is necessary in the cases $\kappa_0 \leq 0$. □

To obtain an estimate of $|\widetilde{K}KL\phi_-|$ we now apply T_0^{-1}, since $\widetilde{K} = T_0^{-1}\widetilde{T}_1$. Let us introduce some notation used in the proposition below. As in Proposition 4, $\chi(x, v; x_0', v_0')$ is the indicator function, which is one when there exist $0 < s < \tau_+(x_0', v_0')$ and $0 < t < \tau_-(x, v)$ such that $\gamma_{(x_0',v_0')}(s) = \gamma_{(x,v)}(t - \tau_-(x, v))$ and equals zero otherwise. In the event that $\chi = 1$, we let $\psi(x, v; x_0', v_0')$ be the angle between the tangent vectors of these geodesics at the point of intersection.

Proposition 7. *For almost every* $(x, v, x_0', v_0') \in \Omega M \times \Gamma_-$ *such that* $\gamma_{(x,v)}(s)$ *and* $\gamma_{(x_0',v_0')}(t)$ *intersect for some* $-\tau_-(x, v) < s < 0$, $0 < t < \tau_+(x_0', v_0')$, *and for almost every* $(x, v, x_0', v_0') \in \Gamma_+ \times \Gamma_-$ *such that* $\gamma_{(x,v)}(s)$ *and* $\gamma_{(x_0',v_0')}(t)$ *intersect for some* $-\tau_-(x, v) < s < 0$, $0 < t < \tau_+(x_0', v_0')$,

$$|\widetilde{K}KL\phi_-(x, v, x_0', v_0')| \leq \begin{cases} \dfrac{\pi A\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}}{\cos(\sqrt{\kappa_0}A)}\left(1 + C_+ + \log\dfrac{C_+'}{|\sin\psi|}\right) & \kappa_0 > 0, \\ A\left(2 + \log\dfrac{2}{|\sin\psi|}\right) & \kappa_0 = 0, \\ A\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\left(1 + C_- + \log\dfrac{C_-'}{|\sin\psi|}\right) & \kappa_0 < 0. \end{cases} \tag{21}$$

where ψ *is evaluated at* $(x, v; x_0', v_0')$, $C_\pm$ *are as in Lemma* 6, $C_+' = \pi$, *and* $C_-' = 2e^{\sqrt{-\kappa_0}A}/\sqrt{-\kappa_0}A$.

Proof. By (8),

$$T_0^{-1}\widetilde{T}_1KL\phi_- = -\int_0^{\tau_-(x,v)} E(\cdot)\widetilde{T}_1KL\phi_-(\vec{\gamma}_{(x,v)}(t - \tau_-(x, v))dt,$$

and from Proposition 5,

$$\begin{aligned} &\widetilde{T}_1KL\phi_-(\vec{\gamma}_{(x,v)}(t - \tau_-(x, v))) \\ &\quad = \int_0^{\tau_+(x_0',v_0')} E(\cdot)E(\cdot)\tilde{k}(\cdot)k(\cdot)\left|X_{\hat{v}}(d(\gamma_{(x,v)}(t - \tau_-(x, v)), \gamma_{(x_o',v_0')}(s))\right|^{-1} ds, \end{aligned}$$

where $\hat{v} = \hat{v}(s, \gamma_{(x,v)}(t - \tau_-(x, v)))$ as in (18), and $X_{\hat{v}}$ is the Jacobi field along the geodesic joining $\gamma_{(x,v)}(t - \tau_-(x, v))$ to $\gamma_{(x_0',v_0')}(s)$ as described in the paragraph below (18). We shall not need the precise arguments of $\tilde{k}$ and k. From the estimate (19) we thus have

$$|T_0^{-1}\widetilde{T}_1KL\phi_-(x, v)| \leq \begin{cases} \dfrac{\pi\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}}{\cos(\sqrt{\kappa_0}A)}\displaystyle\int_0^{\tau_-(x,v)}\left(C_+ + \log\dfrac{A}{v(t)}\right)dt & \kappa_0 > 0, \\ \|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\displaystyle\int_0^{\tau_-(x,v)}\left(1 + \log\dfrac{A}{v(t)}\right)dt & \kappa_0 = 0, \\ \|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\displaystyle\int_0^{\tau_-(x,v)}\left(C_- + \log\dfrac{A}{v(t)}\right)dt & \kappa_0 < 0 \end{cases} \tag{22}$$

where $v(t)$ is the minimum distance between $\gamma_{(x,v)}(t - \tau_-(x, v))$ and the geodesic $\gamma_{(x_0',v_0')}(s)$, $0 \leq s \leq \tau_+(x_0', v_0')$.

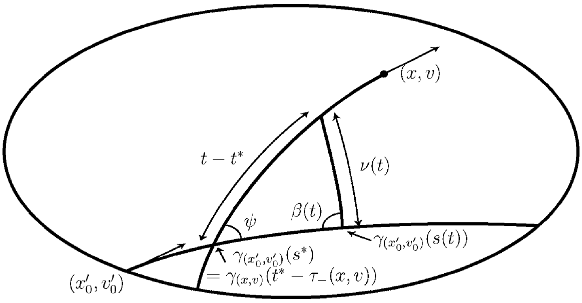

Figure 1. The geodesic triangle used in the proof of Proposition 7.

Let $0 \le s^* \le \tau_+(x_0', v_0')$ and $0 \le t^* \le \tau_-(x, v)$ be such that $\gamma_{(x_0',v_0')}(s^*) = \gamma_{(x,v)}(t^* - \tau_-(x, v))$. Let $s(t)$ be such that $\gamma_{(x_0',v_0')}(s(t))$ is the closest point to $\gamma_{(x,v)}(t)$ (achieving the distance $\nu(t)$). We consider the geodesic triangle formed by the points $\gamma_{(x_0',v_0')}(s^*) = \gamma_{(x,v)}(t^* - \tau_-(x, v))$, $\gamma_{(x_0',v_0')}(s(t))$ and $\gamma_{(x,v)}(t - \tau_-(x, v))$; let $\beta(t)$ be the angle at $\gamma_{(x_0',v_0')}(s(t))$. See Figure 1.

First suppose that $\kappa_0 > 0$. Then $\nu(t)$ is not less than the length of the corresponding side of the comparison triangle on the 2-sphere of radius $1/\sqrt{\kappa_0}$, with the same angle ψ and the same adjacent side-lengths. If β_{κ_0} is the comparison angle to $\beta(t)$ then from the spherical law of sines we thus obtain

$$\frac{\sin\psi(x, v; x_0', v_0')}{\sin(\sqrt{\kappa_0}\,\nu(t))} \le \frac{\sin(\beta_{\kappa_0})}{|\sin(\sqrt{\kappa_0}(t - t^*))|} \le \frac{1}{|\sin(\sqrt{\kappa_0}(t - t^*))|}.$$

For $0 \le |t - t^*| \le A < \pi/2\sqrt{\kappa_0}$, and similarly for $\nu(t)$,

$$\sin(\sqrt{\kappa_0}\,\nu(t)) \le \sqrt{\kappa_0}\,\nu(t) \qquad \text{and} \qquad |\sin(\sqrt{\kappa_0}(t - t^*))| \ge \frac{2\sqrt{\kappa_0}}{\pi}|t - t^*|,$$

so that

$$\frac{A}{\nu(t)} \le \frac{\pi A}{2|t - t^*||\sin\psi(x, v; x_0', v_0')|}$$

and (22) becomes

$$\begin{aligned} |T_0^{-1}\widetilde{T}_1 KL\phi_-(x, v)| \le {} & \frac{\pi}{\cos(\sqrt{\kappa_0}A)}\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty} \\ & \times \int_0^A \left(C_+ + \log\frac{A\pi}{2|t - t^*||\sin\psi(x, v; x_0', v_0')|}\right)dt. \end{aligned} \tag{23}$$

Now

$$\begin{aligned} & \int_0^A \left(C_+ + \log\frac{A\pi}{2|t - t^*||\sin\psi|}\right)dt \\ & = \left(\int_0^{t^*} + \int_0^{A-t^*}\right)\left(C_+ + \log\frac{A\pi}{2t|\sin\psi|}\right)dt \\ & = (1 + C_+)A + (A - t^*)\log\frac{A\pi}{2(A - t^*)|\sin\psi|} + t^*\log\frac{A\pi}{2t^*|\sin\psi|} \\ & \le A\left(1 + C_+ + \log\frac{\pi}{|\sin\psi|}\right) \end{aligned}$$

since the expression is majorized at $t^* = A/2$. Combining this with (23) gives (21) in the case $\kappa_0 > 0$. For $\kappa_0 < 0$, we apply the hyperbolic law of sines and the estimates $\sinh(z) \geq z$ and, for $0 \leq z \leq A$, $\sinh(z) \leq (e^A/A)z$ to obtain

$$\frac{A}{v(t)} \leq \frac{e^{\sqrt{-\kappa_0}A}}{\sqrt{-\kappa_0}|t - t^*||\sin\psi(x, v; x_0', v_0')|}.$$

As in the case for $\kappa_0 > 0$, this yields (21) for $\kappa_0 < 0$. Finally, if $\kappa_0 = 0$, $v(t) \geq |t - t^*||\sin\psi|$, and upon integrating with respect to t as above we obtain (21) for $\kappa_0 = 0$. □

We shall also need the following estimate.

Proposition 8. *We have* $K^3\phi_0 \in L^\infty(\Omega M \times \Gamma_-)$ *and* $K^3\phi_0 \in L^\infty(\Gamma_+ \times \Gamma_-)$ *with norms bounded by* $C\|k\|_{L^\infty}^3$, *where* C *depends only on* (M, g).

Proof. We present the proofs in the cases $\kappa_0 \neq 0$; the case $\kappa_0 = 0$ is handled in the same manner of comparison with flat Euclidean space as demonstrated in previous proofs. Since $K^3\phi_0 = T_0^{-1}T_1(T_0^{-1}T_1KL\phi_-)$, from Lemma 6 we have

$$|K^3\phi_0(x, v, x_0', v_0')| \leq \widehat{C}_\pm \int_0^{\tau_-(x,v)} \int_{\Omega_{y(t)}M} |k(\cdot)| \int_0^{\tau_-(y(t),w)} \left(C_\pm + \log\frac{A}{v(s)}\right) ds\, dw\, dt, \tag{24}$$

where $\widehat{C}_+ = \pi\|k\|_{L^\infty}^2/\cos(\sqrt{\kappa_0}A)$, $\widehat{C}_- = \|k\|_{L^\infty}^2$, $y(t) = \gamma_{(x,v)}(t - \tau_-(x, v))$, and $v(s)$ is the distance from $\gamma_{(y(t),w)}(s - \tau_-(y(t), w))$ to the geodesic $\gamma_0(r) = \gamma_{(x_0',v_0')}(r)$, $0 \leq r \leq \tau_+(x_0', v_0')$.

Fix t. For the time being denote $y(t)$ simply by y. Consider the following geodesics: $\gamma_0(\cdot) = \gamma_{(x_0',v_0')}(\cdot)$; $\gamma_1(\cdot)$ is the geodesic from $\gamma_{(x_0',v_0')}(0)$ through y described by $\gamma_{(y,v_1)}(s)$, $-\tau_-(y, v_1) \leq s \leq \tau_+(y, v_1)$, thus defining $v_1 \in \Omega_yM$; $\gamma_2(\cdot)$ is the geodesic from $\gamma_{(x_0',v_0')}(\tau_+(x_0', v_0'))$ through y described by $\gamma_{(y,v_2)}(s)$, $-\tau_-(y, v_2) \leq s \leq \tau_+(y, v_2)$, thus defining v_2. Choose coordinates $\theta \in [0, 2\pi)$ for Ω_yM such that $\theta(v_1) = 0$ and $0 < \theta(v_2) < \pi$. We compute the integral

$$\int_0^{2\pi} \int_0^{R(-\theta)} \left(C_\pm + \log\frac{A}{d(\exp_y(-r\theta), \gamma_0)}\right) dr\, d\theta, \tag{25}$$

where $R(-\theta) = \tau_-(y, \theta)$ and $d(w, \gamma_0)$ is the geodesic distance from w to γ_0. Of course, by θ we mean the vector in Ω_yM corresponding to the coordinate θ. We split the integral into three pieces (two of which overlap):

$$0 \leq \theta \leq \theta(v_2), \qquad \theta(v_2) \leq \theta \leq \theta(v_2) + \pi, \qquad \pi \leq \theta \leq 2\pi.$$

Let $0 \leq \theta \leq \theta(v_2)$ and suppose that $\kappa_0 > 0$. The geodesic $\gamma_{(y,-\theta)}(r)$ intersects γ_0 at, say, $r = R_0$. For $0 < r \leq R_0$ consider the geodesic triangle $(a, b, c) = (\gamma_{(y,-\theta)}(r), \gamma_{(y,-\theta)}(R_0) = \gamma_{(x_0',v_0')}(s), c)$, where c is the point on γ_0 closest to a. Denote by $\beta(\theta)$ the angle at the vertex b. See Figure 2. Now let $(\tilde{a}, \tilde{b}, \tilde{c})$ be the comparison triangle on the 2-sphere of radius $1/\sqrt{\kappa_0}$, which has angle $\beta(\theta)$ at $\tilde{b}$ and has adjacent sides lengths $d_{\kappa_0}(\tilde{b}, \tilde{c}) = d(b, c)$ and $d_{\kappa_0}(\tilde{b}, \tilde{a}) = d(b, a)$. Then $d(a, c) \geq d_{\kappa_0}(\tilde{a}, \tilde{c})$.

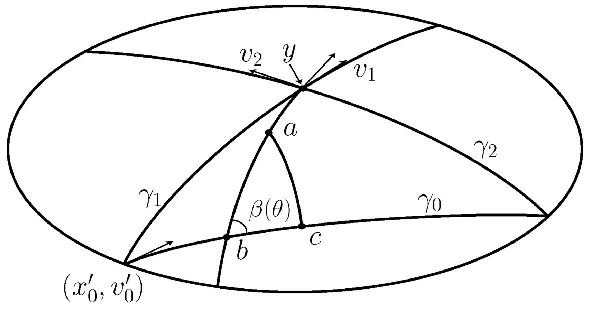

Figure 2. The geodesic triangle (a, b, c) in the proof of Proposition 8.

If $\tilde{\mu}(\theta)$ is the angle at $\tilde{c}$, then from the law of sines on the sphere we have

$$\frac{\sin(\sqrt{\kappa_0}\, d_{\kappa_0}(\tilde{a}, \tilde{c}))}{\sin\beta(\theta)} = \frac{\sin(\sqrt{\kappa_0}(R_0 - r))}{\sin(\tilde{\mu}(\theta))} \geq \sin(\sqrt{\kappa_0}(R_0 - r))$$

so that

$$d(a, c) \geq \frac{1}{\sqrt{\kappa_0}} \sin(\sqrt{\kappa_0}\, d_{\kappa_0}(\tilde{a}, \tilde{c})) \geq \frac{2}{\pi}(R_0 - r)\sin\beta(\theta). \tag{26}$$

For $R_0 \leq r \leq \tau_-(y, \theta)$ the analogous argument gives the same estimate (26). Thus for (25) we obtain

$$\begin{aligned} &\int_0^{R_0} + \int_{R_0}^{R(-\theta)} \left(C_+ + \log\frac{A}{d(\exp_y(-r\theta), \gamma_0)}\right) dr \\ &\quad \leq 2\int_0^A \left(C_+ + \log\frac{\pi A}{2r\sin\beta(\theta)}\right) dr = 2A\left(1 + C_+ + \log\frac{C'_+}{2\sin\beta(\theta)}\right), \end{aligned} \tag{27}$$

and this holds for $0 \leq \theta \leq \theta(v_2)$. When $\kappa_0 < 0$, we apply the hyperbolic law of sines as in Proposition 7 to obtain the estimate (27) with C_+ and C'_+ replaced by C_- and C'_-.

Next, let $\theta(v_2) \leq \theta \leq \theta(v_2) + \pi$. Note that $\gamma_2(t)$, $-\tau_-(y, v_2) \leq t \leq \tau_+(y, v_2)$ divides M into two regions with γ_0 in one region and $\gamma_{(y,\theta)}(s)$, $-\tau_-(y, \theta) \leq s \leq 0$ in the other. For any $0 \leq r \leq \tau_-(y, \theta)$ let $a = \gamma_{(y,-\theta)}(r)$, b be the closest point on γ_0 to a, c be the point of intersection of the geodesic from a to b with γ_2, and d be the closest point on γ_2 to a. See Figure 3. Then we have

$$d(a, b) = d(a, c) + d(c, b) \geq d(a, c) \geq d(a, d).$$

Consider the geodesic triangle (y, a, d) and suppose that $\kappa_0 > 0$; comparing this with the corresponding triangle $(\tilde{y}, \tilde{a}, \tilde{d})$ on the sphere of radius $1/\sqrt{\kappa_0}$ with the same angle at y and adjacent side lengths, we obtain

$$\begin{aligned} d(a, d) &\geq \frac{1}{\sqrt{\kappa_0}} \sin(\sqrt{\kappa_0}\, d_{\kappa_0}(\tilde{a}, \tilde{d})) \\ &= \frac{1}{\sqrt{\kappa_0}} |\sin(\theta - \theta(v_2))| \sin(\sqrt{\kappa_0}\, r) \geq \frac{2}{\pi}\, r\, |\sin(\theta - \theta(v_2))|. \end{aligned}$$

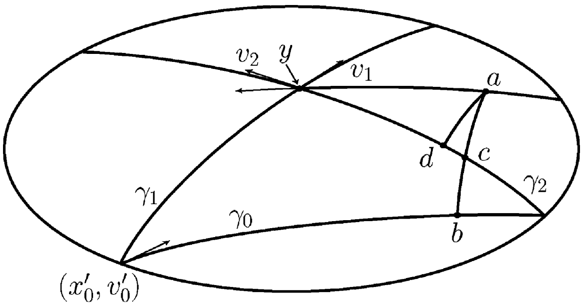

Figure 3. The geometry in the case that $\theta(v_2) \le \theta \le \theta(v_2) + \pi$.

Thus, for $\theta(v_2) \le \theta \le \theta(v_2) + \pi$ and for $\kappa_0 > 0$,

$$\int_0^{R(-\theta)} \left(C_+ + \log \frac{A}{d(\exp_y(-r\theta), \gamma_0)} \right) dr \le A\left(1 + C_+ + \log \frac{C_+'}{2|\sin(\theta - \theta(v_2))|} \right). \quad (28)$$

When $\kappa_0 < 0$ we have that

$$d(a, d) \ge \frac{\sqrt{-\kappa_0}A}{e^{\sqrt{-\kappa_0}A}} r\,|\sin(\theta - \theta(v_2))|$$

which yields (28), with C_+ and C_+' replaced by C_- and C_-'.

The analogous argument for $\pi \le \theta \le 2\pi$ with γ_2 in place of γ_1 gives the same estimate as (28) (and its equivalent for $\kappa_0 < 0$), with $\theta(v_1) = 0$ in place of $\theta(v_2)$.

We now integrate with respect to θ. First, for $0 \le \theta \le \theta(v_2)$ we have

$$\int_0^{\theta(v_2)} 2A\left(1 + C_\pm + \log \frac{C_\pm'}{2 \sin \beta(\theta)} \right). \quad (29)$$

We wish to change variables to integrate with respect to β and so must understand $d\beta/d\theta$. For fixed θ, let $R_0(\theta)$ and $s(\theta)$ be such that

$$\gamma_{(y,\theta)}(-R_0(\theta)) = \gamma_{(x_0', v_0')}(s(\theta));$$

let J be the Jacobi field along $\gamma_{(y,\theta)}(-s)$, $0 \le s \le R_0 = R_0(\theta)$ with $J(0) = 0$, and $\dot{J}(0) = \theta^\perp$ (either choice of $\theta^\perp$ will suffice). Then

$$\cos(\beta(\theta)) = \langle \dot{\gamma}_{(y,\theta)}(-R_0(\theta)), \dot{\gamma}_{(x_0', v_0')}(s(\theta)) \rangle,$$

so

$$-\sin(\beta(\theta)) \frac{d\beta}{d\theta} = \left\langle \frac{d}{d\sigma}\bigg|_{\sigma=\theta} \dot{\gamma}_{(y,\sigma)}(-R_0(\theta)) + \frac{d}{d\sigma}\bigg|_{\sigma=\theta} \dot{\gamma}_{(y,\theta)}(-R_0(\sigma)), \dot{\gamma}_{(x_0', v_0')}(s(\theta)) \right\rangle$$
$$+ \left\langle \dot{\gamma}_{(y,\theta)}(-R_0(\theta)), \ddot{\gamma}_{(x_0', v_0')}(s(\theta)) \frac{ds}{d\theta} \right\rangle$$

$$= \left\langle \frac{d}{d\sigma}\bigg|_{\sigma=\theta} \dot\gamma_{(y,\sigma)}(-R_0(\theta)), \dot\gamma_{(x_0',v_0')}(s(\theta)) \right\rangle$$
$$= \left\| \frac{d}{d\sigma}\bigg|_{\sigma=\theta} \dot\gamma_{(y,\sigma)}(-R_0(\theta)) \right\| \cos\left(\frac{\pi}{2} - \beta(\theta)\right).$$

Now assumption M1, that there are no focal points, implies the existence of $\varepsilon_0 > 0$ such that $\|\dot J(\cdot)\| \geq \varepsilon_0$ uniformly over all Jacobi fields along geodesics of (M, g) satisfying $J(0) = 0$, $\|\dot J(0)\| = 1$. Thus

$$\left|\frac{d\beta}{d\theta}\right| = \|\dot J(R_0(\tau))\| \geq \varepsilon_0.$$

The integral (29) becomes

$$\int_{\beta(0)}^{\beta(\theta(v_2))} 2A\left(1 + C_\pm + \log\frac{C_\pm'}{2\sin\beta}\right)\left|\frac{d\theta}{d\beta}\right| d\beta \leq \frac{2A}{\varepsilon_0}\int_0^\pi \left(1 + C_\pm + \log\frac{C_\pm'}{2\sin\beta}\right) d\beta$$
$$= \frac{2\pi A}{\varepsilon_0}(1 + C_\pm + \log C_\pm'). \tag{30}$$

Integrating (28) over $\theta(v_2) \leq \theta \leq \theta(v_2) + \pi$ and the analogous expression for $\pi \leq \theta \leq 2\pi$ we obtain that the integral (25) is bounded above by

$$2\pi A\left(1 + \frac{1}{\varepsilon_0}\right)(1 + C_\pm + \log C_\pm').$$

Referring back to (24), we integrate with respect to t to obtain that for almost every $(x, v, x_0', v_0') \in \Omega M \times \Gamma_-$ and for almost every $(x, v, x_0', v_0') \in \Gamma_+ \times \Gamma_-$,

$$|K^3\phi_0(x, v, x_0', v_0')| \leq \widetilde C_\pm \|k\|_{L^\infty}^3 (1 + C_\pm + \log C_\pm'),$$

where

$$\widetilde C_\pm = \begin{cases} \dfrac{2\pi^2 A^2}{\cos(\sqrt{\kappa_0}A)}\left(1 + \dfrac{1}{\varepsilon_0}\right) & \kappa_0 > 0, \\ 2\pi A^2\left(1 + \dfrac{1}{\varepsilon_0}\right) & \kappa_0 < 0. \end{cases} \qquad \square$$

5. Proof of Theorem 1

We complete the proof of Theorem 1 after first deriving the estimates of Lemmas 9 and 10. In what follows, C_+ and C_+' are the constants in the case $\kappa_0 > 0$; C_- and C_-' are for the case $\kappa_0 < 0$; when $\kappa_0 = 0$, $C_\pm = 1$ and $C_\pm' = 2$.

Lemma 9. *For almost every* $(x, v, x_0', v_0') \in \Gamma_+ \times \Gamma_-$ *such that* $\gamma_{(x,v)}(t) = \gamma_{(x_0',v_0')}(s)$ *for some* $-\tau_-(x, v) < t < 0$ *and* $0 < s < \tau_+(x_0', v_0')$, *we have*

$$|(I + K)^{-1}(K^2 - \widetilde K^2)\phi_0(x, v, x_0', v_0')|$$
$$\leq C\|k - \tilde k\|_{L^\infty}(\|k\|_{L^\infty} + \|\tilde k\|_{L^\infty})\left(1 + C_\pm + \log\frac{C_\pm'}{|\sin\psi|}\right)$$

where C *depends only on* (M, g), *and* ψ *is the angle of intersection of the geodesics.*

Proof. Write

$$\begin{aligned}(I+K)^{-1}(K^2-\widetilde{K}^2) &= \big(I-(I+K)^{-1}K\big)\big(K(K-\widetilde{K})+(K-\widetilde{K})\widetilde{K}\big)\\ &= K(K-\widetilde{K})+(K-\widetilde{K})\widetilde{K}-(I+K)^{-1}K^2(K-\widetilde{K})\\ &\quad +(I+K)^{-1}K(K-\widetilde{K})\widetilde{K}.\end{aligned}$$

The contribution from the first two terms is estimated by (21) from Proposition 7 with either K or $\widetilde{K}$ replaced by $K-\widetilde{K}$ (the proof proceeds unaltered). For the final two terms, $(K^2(K-\widetilde{K})+K(K-\widetilde{K})\widetilde{K})\phi_0$ is a function in L^∞ by Proposition 8, and then $(I-K)^{-1}$ preserves this space. $\square$

Lemma 10. *For almost every* $(x, v, x'_0, v'_0) \in \Gamma_+ \times \Gamma_-$ *such that* $\gamma_{(x,v)}(t) = \gamma_{(x'_0,v'_0)}(s)$ *for some* $-\tau_-(x, v) < t < 0$ *and* $0 < s < \tau_+(x'_0, v'_0)$, *we have*

$$|(I+\widetilde{K})^{-1}(\widetilde{K}-K)(I+K)^{-1}\widetilde{K}^2\phi_0(x, v, x'_0, v'_0)| \le C\|k-\tilde{k}\|_{L^\infty}\|\tilde{k}\|^2_{L^\infty}(1+\|k\|_{L^\infty})$$

where C *depends only on* (M, g).

Proof. Writing $(I+K)^{-1}\widetilde{K}^2 = \widetilde{K}^2-(I+K)^{-1}K\widetilde{K}^2$ and applying Proposition 8 as in the previous lemma yields the claim. $\square$

Proof of Theorem 1. By (15),

$$\chi(\alpha_2-\tilde{\alpha}_2)(x, v, x', v') = \chi|E(\cdot)E(\cdot)\mathcal{J}(\cdot)|\frac{|(\tilde{k}-k)(y, w, \hat{w})|}{|\sin\psi|} \qquad \text{a.e.,}$$

so that

$$\chi|(\tilde{k}-k)(y, w, \hat{w})| \le C_1 e^{2A\Sigma}\chi|\sin\psi||(\alpha_2-\tilde{\alpha}_2)(x, v, x', v')| \qquad \text{a.e.,} \tag{31}$$

where C_1 is a uniform constant depending only on (M, g), y is the point of intersection $y = \gamma_{(x',v')}(s) = \gamma_{(x,v)}(t-\tau_-(x, v))$, $w = \dot{\gamma}_{(x',v')}(s) \in \Omega_y M$, and $\hat{w} = \dot{\gamma}_{(x,v)}(t-\tau_-(x, v)) \in \Omega_y M$. Recall that $\psi = \psi(x, v, x', v')$ is the angle between w and $\hat{w}$ at y, and that $\|\sigma_a\|_{L^\infty} \le \Sigma$ (see (2)). Referring back to (17) and applying Lemmas 9 and 10, we also have

$$\chi(\alpha_2-\tilde{\alpha}_2)(x, v, x', v') \le C_2\varepsilon\|k-\tilde{k}\|_{L^\infty}\left(1+\log\frac{C'_\pm}{|\sin\psi|}\right) \qquad \text{a.e.,}$$

so that

$$C_1 e^{2A\Sigma}\chi|\sin\psi||(\alpha_2-\tilde{\alpha}_2)(x, v, x', v')| \le C_3 e^{2A\Sigma}\varepsilon\|k-\tilde{k}\|_{L^\infty} \qquad \text{a.e.,} \tag{32}$$

where again C_3 depends only on (M, g). Combining (31) and (32) we obtain

$$\|k-\tilde{k}\|_{L^\infty(X)} \le C_3 e^{2A\Sigma}\varepsilon\|k-\tilde{k}\|_{L^\infty(X)},$$

where $X = \{(y, w, \hat{w}) \in M \times \Omega_y M \times \Omega_y M\}$. The factor χ appearing on the left-hand side of (31) no longer appears in the final estimate, since χ is identically one on the space X. Thus, for sufficiently small ε, we have $k = \tilde{k}$, and we may take $\varepsilon = Ce^{-2A\Sigma}$ with C depending only on (M, g) as claimed. ☐

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References

Arridge, S. (1999). Optical tomography in medical imaging. *Inverse Problems* 15:R41–R93.

Choulli, M., Stefanov, P. (1996). Inverse scattering and inverse boundary value problems for the linear Boltzmann equation. *Comm. Partial Differential Equations* 21(5–6):763–785.

Choulli, M., Stefanov, P. (1999). An inverse boundary value problem for the stationary transport equation. *Osaka J. Math.* 36(1):87–104.

Ferwerda, H. A. (1999). The radiative transfer equation for scattering media with a spatially varying refractive index. *J. Opt. A: Pure Appl. Opt.* 1:L1–L2.

Helgason, S. (1978). *Differential Geometry, Lie Groups and Symmetric Spaces*. New York: Academic Press.

Katok, A., Hasselblatt, B. (1995). *Introduction to the Modern Theory of Dynamical Systems*. Cambridge: Cambridge University Press.

McDowall, S. (2004). An inverse problem for the transport equation in the presence of a Riemannian metric. *Pac. J. Math.* 216(2):303–326.

Paternain, G. (1999). *Geodesic Flows*. Progress in Mathematics. Boston: Birkhauser.

Pestov, L., Uhlmann, G. (to appear). Two dimensional simple Riemannian manifolds are boundary distance rigid. *Anals of Math.*

Pestov, L., Uhlmann, G. (2004). The boundary distance function and the Dirichlet-to-Neumann map. *Math. Res. Lett.* 11(2–3):285–297.

Reed, M., Simon, B. (1979). *Methods of Modern Mathematical Physics, III: Scattering Theory*. New York: Academic Press.

Romanov, V. G. (1996). Stability estimates in problems of recovering the attenuation coefficient and the scattering indicatrix for the transport equation. *J. Inverse Ill-Posed Probl.* 4(4):297–305.

Sharafutdinov, V. A. (1994). *Integral Geometry of Tensor Fields*. Inverse and Ill-Posed Problems Series. Utrecht, The Netherlands: VSP.

Sharafutdinov, V. A. (1999). The inverse problem of determining the source in the stationary transport equation on a Riemannian manifold. *J. Math. Sci.* (New York) 96(4): 3430–3433.

Stefanov, P., Uhlmann, G. (2003). Optical tomography in two dimensions. *Methods Appl. Anal.* 10(1):1–9.

Tamasan, A. (2002). An inverse boundary value problem in two-dimensional transport. *Inverse Problems* 18(1):209–219.

Tamasan, A. (2003). Optical tomography in weakly anisotropic scattering media. *Inverse Problems: Theory and Applications (Cortona/Pisa, 2002).* Contemp. Math. 333. Providence, RI: Amer. Math. Soc., pp. 199–207.

Wang, J. N. (1999). Stability estimates of an inverse problem for the stationary transport equation. *Ann. Inst. H. Poincaré Phys. Théor.* 70(5):473–495.