Give detailed explanations for your answers. There are four problems. Each is worth 25 points.

1. Figure 1 shows an oscillating string. The equation of the oscillating string is \( y = F(x, t) = (\sin x)(\cos t) \) where \( x \in [0, \pi) \), \( t \geq 0 \). Here, for a fixed time \( t = t_0 \), \( y = F(x, t_0) \) describes the shape of the string at time \( t_0 \). In Figure 1 the string at time \( t = t_0 \) is black. To indicate the motion of the string, I added several previous positions of the string in various shade of gray. Consider the following seven quantities:

\[
F(x_0, t_0), \quad F_x(x_0, t_0), \quad F_t(x_0, t_0), \quad F_{xx}(x_0, t_0), \quad F_{xt}(x_0, t_0), \quad F_{tx}(x_0, t_0), \quad F_{tt}(x_0, t_0).
\]

Based on Figure 1 for each of the seven quantities listed above determine whether it is positive or negative.

![Figure 1: An oscillating string](image)

2. The goal of this problem is to make the cheapest storage box with a fixed volume, as shown in Figure 2. For simplicity we can assume that the fixed volume is 1 cubic unit. As you can see in Figure 2 the storage box is build on a side of a house. It has three vertical “walls” made of chain-link fencing and the roof. The roofing material costs three times as much (per square unit) as chain-link fencing. Find the dimensions (depth, width and height) of the storage box that will minimize the cost of the materials.

Give both: exact and approximate values for the dimensions of the box.

Use the **second derivative test** to confirm that the point you obtained is a local minimum.

![Figure 2: A storage](image)

3. Consider the function \( F(x, y) = 4x\sqrt{y} - 4\ln(xy) \). (You can think of \( F \) as being a temperature at each point of a heated plate.) Consider the point \( P = (4, 1) \).

(a) Find the vector in the direction of maximum rate of change of \( F \) at the point \( P \). What is the maximum rate of change of \( F \)?

(b) Find the instantaneous rate of change of \( F \) as you leave \( P \) heading toward the point \( (2, 3) \).

(c) Find a vector in a direction in which the rate of change of \( F \) at \( P \) is 0.

4. Consider the hyperboloid \( x^2 + y^2 - z^2 = 1 \). Is there a point on this hyperboloid at which the tangent plane to the hyperboloid is parallel to the plane \( x + y + z = 0 \)? If so, find it, if not explain why not. If there is more than one such point find all of them.
1. \( F(x_0, t_0) < 0 \) position is below x-axis
   \( F_x(x_0, t_0) > 0 \) the slope of the string is >0
   \( F_t(x_0, t_0) > 0 \) the string's position is increasing

   \( F_{xx}(x_0, t_0) > 0 \) the string is waving.
   \( F_{xt}(x_0, t_0) < 0 \) the slope is decreasing.
   \( F_{tx}(x_0, t_0) < 0 \) the velocity is decreasing with increasing x
   \( F_{tt}(x_0, t_0) > 0 \) the string is speeding up.

2. \( 3xy + 2xz + yz \rightarrow \) is the cost of material
   \( xyzt = 1 \) volume
   \( z = \frac{1}{xy} \)
   \[ C(x, y) = 3xy + \frac{2}{y} + \frac{1}{x} \]
2) Find CR-s:

\[ \begin{align*}
  C_x &= 3y - \frac{1}{x^2} = 0 \\
  C_y &= 3x - \frac{2}{y^2} = 0
\end{align*} \]

Solve for \( x, y > 0 \).

\[ y = \frac{1}{3x^2} \Rightarrow 3x - \frac{2}{9x^4} = 0 \]

\[ 6x^3 = 1 \Rightarrow x = \frac{1}{3} \sqrt[3]{6} \]

\[ y = \frac{1}{3} \text{ in } 6^{2/3} = \frac{6^{2/3}}{3} = \frac{2}{3} = \frac{2^{2/3}}{3^{1/3}} = \sqrt[3]{4/3} \]

\[ z = \frac{1}{3} \Rightarrow \frac{1}{z^{3/2}} = \frac{1}{3^{3/2}} = \frac{1}{3} \sqrt[3]{6} \]

\[ \frac{1}{z^{3/2}} \times y = \frac{1}{3^{3/2}} \times \frac{1}{3x^2} = \frac{1}{3} \sqrt[3]{6} \]

The second derivative test

\[ C_{xx} = \frac{2}{x^3}, \quad C_{xy} = 3, \quad C_{yy} = \frac{4}{y^3} \]

\[ D = \frac{8}{(xy)^3} - 9 = \frac{8}{27} \cdot 9 - 9 = \frac{36 - 9}{3} > 0 \]
\[ F_x = 4\sqrt{y} - \frac{4}{x} \quad \text{at } (4,1) \]
\[ F_y = \frac{2x}{\sqrt{y}} - \frac{4}{y} \]

(a) \[ (\nabla F)(4,1) = 3\hat{i} + 4\hat{j} \]
\[ \| (\nabla F)(4,1) \| = 5 \]

- direction of max change
- max rate of change

(c) \[ 2\hat{i} + 3\hat{j} - (4\hat{i} + \hat{j}) = -2\hat{i} + 2\hat{j} \]
\[ \mathbf{u} = \frac{1}{\sqrt{2}} (3\hat{i} + \hat{j}) \]
\[ \mathbf{u} \cdot \nabla F = -\frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \]

(c) One such vector is \(-4\hat{i} + 3\hat{j}\) or another one \(4\hat{i} - 3\hat{j}\), orthogonal to \((\nabla F)(4,1)\).
\[ \mathbf{N} = \mathbf{\hat{z}} + \mathbf{\hat{y}} + \mathbf{\hat{e}} \]

Set \( H(x,y,z) = x^2 + y^2 - z^2 \)

(\( \nabla H \))(x,y,z) = 2x\mathbf{\hat{x}} + 2y\mathbf{\hat{y}} - 2z\mathbf{\hat{z}}

Is it possible to find \( \lambda \) such that

\[ 2x\mathbf{\hat{x}} + 2y\mathbf{\hat{y}} = 2z\mathbf{\hat{z}} = \lambda (\mathbf{\hat{x}} + \mathbf{\hat{y}} + \mathbf{\hat{z}}) \]

or

\[ 2x = \lambda \Rightarrow x = \frac{\lambda}{2} \]

\[ 2y = \lambda \Rightarrow y = \frac{\lambda}{2} \]

\[ -2z = \lambda \Rightarrow z = -\frac{\lambda}{2} \]

We need

\[ x^2 + y^2 - z^2 = 1 \]

So

\[ \frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{4} = 1 \]

Thus \( \lambda^2 = 4 \) or \( \lambda = 2 \) or \( \lambda = -2 \)

This gives us two points

\( (1,1,-1) \) and \( (-1,-1,1) \).

At these two points tangent plane to hyperboloid are \( \parallel \) to the given plane.