Stirling Numbers of the 1st Kind

Daniel Reiss, Colebrook Jackson, Brad Dallas

Western Washington University

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The set of all permutations of a set $N$ is denoted $S(N)$, while the set of all permutations of \{1, 2, \ldots, n\} is denoted $S(n)$.

A permutation $\sigma \in S(n)$ is a bijective mapping whose canonical notation is

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}.$$ 

We call $\sigma = \sigma(1) \sigma(2) \ldots \sigma(n)$ the word representation of $\sigma$. 
With composition $S(n)$ forms the symmetric group of order $n$. We read a product always from right to left, thus for

$$\sigma = \begin{pmatrix} 123456 \\ 234165 \end{pmatrix}, \quad \tau = \begin{pmatrix} 123456 \\ 134526 \end{pmatrix},$$

we have

$$\tau\sigma = \begin{pmatrix} 123456 \\ 345162 \end{pmatrix}, \quad \text{and} \quad \sigma\tau = \begin{pmatrix} 123456 \\ 241635 \end{pmatrix}.$$  

For example, $3 \to \tau(\sigma(3)) = \tau(4) = 5.$
Another way to describe $\sigma$ is by its cycle decomposition.

For every $i$, the sequence $i, \sigma(i), \sigma^2(i), \ldots$ must eventually terminate with, say, $\sigma^k(i) = i$.

We denote the cycle containing $i$ by $(i, \sigma(i), \sigma^2(i), \ldots, \sigma^{k-1}(i))$.

Repeating this for all elements, we arrive at the cycle decomposition $\sigma = \sigma_1 \sigma_2 \cdots \sigma_t$. 
Example. (i). \( \sigma = \begin{pmatrix} 12345678 \\ 35146827 \end{pmatrix} \) has word representation \( \sigma = 35146827 \) and cycle form \( \sigma = (13)(25678)(4) \).

(ii). \( \sigma = \begin{pmatrix} 12345678 \\ 12354786 \end{pmatrix} \) has word representation \( \sigma = 12354786 \) and cycle form \( \sigma = (1)(2)(3)(45)(678) \).
Cycle Decomposition

- We may start a cycle with any element in the cycle. Thus, (2568) and (8256) are the same cycle.
- Order of cycles is irrelevant.
- Cycles of length 1 are fixed points in \( \sigma \).
- Cycles of length 2 are transpositions \( i \leftrightarrow j \).
- A permutation with only 1 cycle is called cyclic. There are \((n - 1)\) cyclic permutations of \( n \).
Stirling Numbers of the First Kind

The Stirling number $s(n, k)$ of the first kind is the number of permutations of an $n$-set with precisely $k$ cycles.

Define $s(0, 0) = 1$ and $s(0, k) = 0$ for $k > 0$.

Several different notations for the Stirling numbers are in use. Stirling numbers of the first kind are written with a small $s$, and those of the second kind with a large $S$. The Stirling numbers of the second kind are never negative, but those of the first kind can be negative; hence, there is a separate notation for the unsigned Stirling numbers of the first kind.
Common notations for the Stirling numbers include:

- \( s(n, k) = s_{n,k} = \genfrac{[}{]}{0pt}{}{n}{k}(-1)^{n-k} \)
  for the ordinary signed Stirling numbers of the first kind

- \( c(n, k) = \genfrac{[}{]}{0pt}{}{n}{k} = |s(n, k)| \)
  for the unsigned Stirling numbers of first kind

- \( S(n, k) = \genfrac{}{}{0pt}{}{n}{k} = S^{(k)}_n = S_{n,k} \)
  for the Stirling numbers of the second kind

Note: Aigner uses \( s_{n,k} \) for the signless Stirling numbers and for this presentation we will use \( s(n, k) \) for the signless Stirling numbers.
The table lists the first values of the Stirling matrix $s(n, k)$.

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</table>

Stirling numbers of the first kind $s(n, k)$
A useful graphical representation is to interpret $\sigma \in S(n)$ as a directed graph with $i \rightarrow j$ if $j = \sigma(i)$. The following graphs illustrate the Stirling numbers $s(5, k)$ for $k = 1, \ldots, 5$.

$s(5, 1) = 24$ (5 objects with 1 cycle)
$s(5, 2) = 50$  (5 objects with 2 cycles)
\[ s(5, 3) = 35 \quad (5 \text{ objects with } 3 \text{ cycles}) \]
\[ s(5, 4) = 10 \quad \text{(5 objects with 4 cycles)} \]

\[ s(5, 5) = 1 \quad \text{(5 objects with 5 cycles)} \]
The Stirling numbers, $s(n, k)$, satisfy the recurrence relation:

$$s(n, k) = s(n - 1, k - 1) + (n - 1)s(n - 1, k) \quad (n \geq 1)$$

with initial conditions $s(0, 0) = 1$ and $s(n, 0) = s(0, n) = 0$, $n > 0$. 
Proof.

Consider forming a new permutation with $n$ objects from a permutation of $n-1$ objects by adding a distinguished object. There are exactly two ways in which this can be accomplished.

First, we could form a singleton cycle, leaving the extra object fixed. This increases the number of cycles by 1 and so accounts for the $s(n-1, k-1)$ term in the recurrence.

Second, we could insert the object into one of the existing cycles. Consider an arbitrary permutation of $n-1$ objects with $k$ cycles. To form the new permutation, we insert the new object before any of the $n-1$ objects already present. This explains the $(n-1)s(n-1, k)$ term of the recurrence.

These two cases include all of the possibilities, so the recurrence relation follows with the given initial conditions.
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Combinatorics

Stirling Numbers of the 1st Kind
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\[
\begin{align*}
\quad s(n, k) &= s(n - 1, k - 1) + (n - 1)s(n - 1, k) \\
\sum_{k=0}^{n} s(n, k) &= n!
\end{align*}
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- $s(n, k) = s(n - 1, k - 1) + (n - 1)s(n - 1, k)$
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- $s(n, n - 3) = \binom{n}{2}\binom{n}{4}$
Proof.

The recursion has already been established so we prove first that

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$$\sum_{k=0}^{\ell} s(\ell, k) = \ell!.$$  

For $n = \ell + 1$ we have,
Proof.

\[
\sum_{k=0}^{\ell+1} s(\ell + 1, k) = \sum_{k=0}^{\ell+1} \left( s(\ell, k - 1) + \ell s(\ell, k) \right)
\]

\[
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This completes the inductive step and the proof of the claim.
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Proof.

We now show the remaining claims starting with the second which says that $s(n, 1) = (n - 1)!$. 

Consider $s(n, n-1)$. To count these permutations we need only choose which 2 of $\{1, \ldots, n\}$ are going to share a cycle while the others are represented by a singleton cycle. Thus, $s(n, n-1) = \binom{n}{2}$. 

Combinatorics
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$$s(n, 2) = \frac{(n - 2)!}{(n - 1)!} + \frac{(n - 1)s(n - 1, 2)}{(n - 1)!} = s(n - 1, 2) + \frac{1}{n - 1},$$

By repeated iteration we arrive at the following,

$$s(n, 2) = \frac{1}{n - 1} + \frac{1}{n - 2} + \cdots + \frac{1}{2} + 1 = H_{n - 1}.$$

Multiplying by $(n - 1)!$ completes the proof.
Proof.

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$$\frac{s(n, 2)}{(n - 1)!} = \frac{(n - 2)!}{(n - 1)!} + \frac{(n - 1)s(n - 1, 2)}{(n - 1)!} = \frac{s(n - 1, 2)}{(n - 2)!} + \frac{1}{n - 1}.$$
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\[ x^0 = s(0, 0) = 1 \text{ and } x^\bar{1} = s(1, 0) + x s(1, 1) = x. \]
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Now assume the claim is true for \( n = \ell - 1 \). That is,

\[ x^{\ell-1} = \sum_{k=0}^{\ell-1} s(\ell - 1, k)x^k. \]
Proof.

For $n = \ell$, we have

\[
x^\ell = x^{\ell-1} (x + \ell - 1)
= x \cdot x^{\ell-1} + (\ell - 1) x^{\ell-1}
= x \sum_{k \geq 0} s(\ell - 1, k) x^k + (\ell - 1) \sum_{k \geq 0} s(\ell - 1, k) x^k
= \sum_{k \geq 0} s(\ell - 1, k) x^{k+1} + (\ell - 1) \sum_{k \geq 0} s(\ell - 1, k) x^k
= \sum_{k \geq 1} s(\ell - 1, k - 1) x^k + (\ell - 1) \sum_{k \geq 0} s(\ell - 1, k) x^k
= \sum_{k \geq 0} \left[ s(\ell - 1, k - 1) + (\ell - 1)s(\ell - 1, k) \right] x^k
= \sum_{k \geq 0} s(\ell, k) x^k.
\]
Proof.

By the reciprocity law, we have $x^n = (-1)^n(-x)^n$. Thus,

$$x^n = (-1)^n \sum_{k=0}^{n} s(n, k)(-x)^k$$

$$= \sum_{k=0}^{n} (-1)^{n-k} s(n, k)x^k$$

Combinatorics

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Proof.
Next we prove the so called inversion formula.

Let $A$ be an infinite lower triangular matrix. Then $A$ has a unique inverse $A^{-1}$ which is also lower triangular if and only if $A$ has non-zero entries along the main diagonal.

Let $A = (a_{i,j})$ and $A^{-1} = (a_{-1}^{-1}(i,j))$. Then a straightforward calculation shows that,

$$\sum_{n \geq 0} a_{j,n} a_{-1}^{-1}(n,k) = \delta_{jk}.$$ 

Note now that we have the Stirling connection,

$$x^n = \sum_{k \geq 0} S(n,k) x^k \iff x^n = \sum_{k \geq 0} (-1)^{n-k}s(n,k) x^k.$$ 

This statement implies that the matrices for the signed Stirling numbers of the first kind and the Stirling numbers of the second kind are in fact inverses of one another. This together with the previous establishes the claim.
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$$x^n = \sum_{k \geq 0} S(n, k)x^k \iff x^n = \sum_{k \geq 0} (-1)^{n-k}s(n, k)x^k.$$
Proof.

Next we prove the so called inversion formula. Let $A$ be an infinite lower triangular matrix. Then $A$ has a unique inverse $A^{-1}$ which is also lower triangular if and only if $A$ has non-zero entries along the main diagonal. Let $A = (a(i, j))$ and $A^{-1} = (a^{-1}(i, j))$. Then a straightforward calculation shows that,

$$
\sum_{n \geq 0} a(j, n)a^{-1}(n, k) = \delta_{jk}.
$$

Note now that we have the Stirling connection,

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x^n = \sum_{k \geq 0} S(n, k)x^k \iff x^n = \sum_{k \geq 0} (-1)^{n-k}s(n, k)x^k.
$$

This statement implies that the matrices for the signed Stirling numbers of the first kind and the Stirling numbers of the second kind are in fact inverses of one another. This together with the previous establishes the claim.
A variety of identities may be derived by manipulating the generating function. The following result gives us the exponential generating function for the signed Stirling numbers:
A variety of identities may be derived by manipulating the generating function. The following result gives us the exponential generating function for the signed Stirling numbers:

\[
\sum_{n \geq 0} (-1)^{n-k} s(n, k) \frac{z^n}{n!} = \frac{(\log(1 + z))^k}{k!}
\]
Proof.

We utilize one of our many familiar generating functions as a starting point.
Proof.

We utilize one of our many familiar generating functions as a starting point. We have that

\[(1 + z)^m = \sum_{n \geq 0} \binom{m}{n} z^n\]

\[= \sum_{n \geq 0} \frac{m^n}{n!} z^n\]

\[= \sum_{n \geq 0} \frac{z^n}{n!} \sum_{k \geq 0} (-1)^{n-k} s(n, k) m^k\]

\[= \sum_{k \geq 0} m^k \sum_{n \geq 0} \frac{z^n}{n!} (-1)^{n-k} s(n, k)\]
Proof.
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Now we use the following identity,

\[
(1 + z)^m = e^{m \log(1+z)} = \sum_{k \geq 0} (\log(1 + z))^k \frac{m^k}{k!}
\]

Comparing the two we get the desired result.
Proof.

Now we use the following identity,

\[(1 + z)^m = e^{m \log(1 + z)} = \sum_{k \geq 0} (\log(1 + z))^k \frac{m^k}{k!}\]

Comparing the two we get the desired result. \(\square\)


