Math 566: Homework Set 3

Directions: Write careful and complete solutions to the following problems. You will be graded on your writing as well as your solutions. You are welcome to discuss the problems with others in the class, but you must write your solutions yourself, in your own words. If you receive significant help on a problem, you should give credit. Your work should be typed (preferably in LaTeX) or neatly handwritten. This is due on Wednesday, November 28.

1. Prove that
\[ \sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{k \geq 1} \frac{1 - q^{2k}}{1 - q^{2k-1}}. \]
Interpret this identity combinatorially, as a 'Triangular Number Theorem'.

2. Let \( b_n \) denote the number of (labeled) rooted trees on the vertex set \([n]\) whose leaves are colored either red or blue.

   (a) Find an equation satisfied by the exponential generating function
   \[ B(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!} = 2z + 4z^2 \frac{z^2}{2!} + 24z^3 \frac{z^3}{3!} + \cdots. \]

   (b) Use the Lagrange inversion formula to deduce that
   \[ b_n = \sum_{k=0}^{n} \binom{n}{k} k^{n-1}. \]

Also do the following problems in the text: 3.5, 3.13, 3.32, 3.47, and 3.67.