Srinivasa Ramanujan

(22 December 1887 - 26 April 1920)
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- Education
  - Government Arts College
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Ramanujan's Congruences

Bryan Clark, Lesley Lowery, Nhan Nguyen

Introduction

History
  - 1919
  - 1921

Proof

Extensions
  - 1944
  - Rank example
  - 1954
  - 1984
  - Crank example

Thank you

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\[
\lim_{x \to \infty} N(x) \sqrt{\ln(x)} \approx 0.76422365358922066299069872135
\]

Ramanujan theta function

\[
f(a, b) = \sum_{n=-\infty}^{\infty} \frac{a^n (n+1)}{2 b^n (n-1)}
\]

Rogers–Ramanujan identities

Ramanujan conjecture

Ramanujan’s sum

\[c_q(n) = \sum_{a=1}^{\infty} \left( \frac{a}{q} \right) \frac{e^{2\pi i a q n}}{2}\]
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  - Landau - Ramanujan constant
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    f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}
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  - Rogers - Ramanujan identities
  - Ramanujan conjecture
  - Ramanujan’s sum
    \[
    c_q(n) = \sum_{{a=1 \atop (a,q)=1}}^{q} e^{2\pi i \frac{a}{q} n}
    \]
1919 - Ramanujan first postulates his congruences, inspired when he examined a table of the values of $p(n)$ for values of $n$ from 1 to 200.
The History of Ramanujan’s Congruences

- 1919 - Ramanujan first postulates his congruences, inspired when he examined a table of the values of $p(n)$ for values of $n$ from 1 to 200.
  - Proof of mod 5, mod 7 congruences
  - Uses Theta functions, $f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$, to “sketch” proofs of the congruences:
    
    $p(25n - 1) \equiv 0 \pmod{25}$
    
    $p(49n - 2) \equiv 0 \pmod{49}$. 
1921 - After Ramanujan’s death in 1920, G. H. Hardy edits and publishes one of Ramanujan’s manuscripts which contains new proofs.
The History of Ramanujan’s Congruences

- 1921 - After Ramanujan’s death in 1920, G. H. Hardy edits and publishes one of Ramanujan’s manuscripts which contains new proofs.
  - First published proof of the mod 11 congruence
  - Proofs make use of Eisenstein series
Statement of the Theorem

Theorem

Let \( n \in \mathbb{N} \) then

\[
p(5n + 4) \equiv 0 \pmod{5}
\]
Proof

Proof.

We are looking for $p(5n + 4)$ so we are interested in

$$\left[z^{5n+4}\right] \sum_{n \geq 0} p(n)z^n$$

So it suffices to look at

$$\left[z^{5n}\right] \left(\frac{z}{\prod_{i \geq 1}(1 - z^i)}\right)$$  \hspace{1cm} (*)
To get at star, look at

$$z \cdot \left( \prod_{i \geq 1} (1 - z^i) \right)^3 \cdot \prod_{i \geq 1} (1 - z^i)$$

Through some algebra we find that this equals:

$$\sum_{r \geq 0} \sum_{s \in \mathbb{Z}} (-1)^r + s (2r + 1) z^{1 + \frac{r(r+1)}{2}} + \frac{s(3s+1)}{2}$$

(◇)
Continued.

If the exponent of $\diamondsuit$ is a multiple of 5

$$1 + \frac{r(r + 1)}{2} + \frac{s(3s + 1)}{2} \equiv 0 \pmod{5}$$

We see that

$$(2r + 1)^2 + 2(s + 1)^2 \equiv 0 \pmod{5}$$

Therefore

$$(2r + 1) \equiv 0 \pmod{5}$$
Continued.

From the last slide

\[ [z^{5n}] \left( z \left( \prod_{i \geq 1} (1 - z^i) \right)^4 \right) \equiv 0 \pmod{5} \quad (** \right)
Continued.

To get (\(*\)) let’s now consider

\[
[z^{5n}] \left( \frac{z}{\Pi_{i \geq 1} (1 - z^i)} \cdot \Pi_{i \geq 1} (1 - z^{5i}) \right).
\]

We multiply by 1,

\[
[z^{5n}] \left( z \left( \prod_{i \geq 1} (1 - z^i) \right)^4 \cdot \frac{\Pi_{i \geq 1} (1 - z^{5i})}{(\Pi_{i \geq 1} ((1 - z^i)^5))} \right).
\]
Continued.

By binomial expansion

\[(1 - z^i)^5 \equiv (1 - z^{5i}) \pmod{5}\]

\[\prod_{i \geq 1} (1 - z^i)^5 \equiv \prod_{i \geq 1} (1 - z^{5i}) \pmod{5}\]

\[\frac{\prod_{i \geq 1} (1 - z^{5i})}{\prod_{i \geq 1} (1 - z^i)^5} \equiv 1 \pmod{5}.\]
From \( (∗∗) \) and \( (∗∗) \) we have

\[
\left[ z^{5n} \right] \left( z \left( \prod_{i \geq 1} (1 - z^i) \right)^4 \cdot \frac{\prod_{i \geq 1} (1 - z^{5i})}{\prod_{i \geq 1} ((1 - z^i)^5)} \right) \equiv 0 \pmod{5}
\]

\[
\left[ z^{5n} \right] \left( \frac{z}{\prod_{i \geq 1} (1 - z^i)} \cdot \prod_{i \geq 1} (1 - z^{5i}) \right) \equiv 0 \pmod{5}
\]
Continued.

Finally

\[
\left[ z^{5n} \right] \left( \frac{z}{\prod_{i \geq 1} (1 - z^i)} \right) \quad (*)
\]
Theorem

Let $n \in \mathbb{N}$ then

$$p(7n + 5) \equiv 0 \pmod{7}$$
Proof.

Similar to the \((\text{mod } 5)\) proof
(mod 11) Case

Theorem

\[ p(11n + 6) \equiv 0 \pmod{11} \]
We will show that the style of proof used to prove the \( (\text{mod } 5) \) and the \( (\text{mod } 7) \) cases will not work for the \( (\text{mod } 11) \) case.
We will show that the style of proof used to prove the \((\text{mod } 5)\) and the \((\text{mod } 7)\) cases will not work for the \((\text{mod } 11)\) case.
Attempt.

We are looking for \( p(11n + 6) \) so we are interested in

\[
[z^{11n+6}] \sum_{n \geq 0} p(n)z^n
\]

It would be sufficient to look at

\[
[z^{11n}] \left( \frac{z^5}{\prod_{i \geq 1}(1 - z^i)} \right) \quad (†)
\]
To get at $\dagger$, look at

$$z^5 \left( \prod_{i \geq 1} (1 - z^i) \right)^9 \cdot \prod_{i \geq 1} (1 - z^i)$$

Through some algebra we find that this equals:

$$\sum (-1)^{r+s+t+u} (2r + 1)(2s + 1)(2t + 1)z^\alpha$$

Where $\alpha = 5 + \frac{r(r+1)+s(s+1)+t(t+1)+u(3u+1)}{2}$

The summations extending form $0 < r < \infty$, $0 < s < \infty$, $0 < t < \infty$, and $u \in \mathbb{Z}$
continued.

If the exponent of (*) is a multiple of 11

\[ 5 + \frac{r(r + 1)}{2} + \frac{s(s + 1)}{2} + \frac{t(t + 1)}{2} + \frac{u(3u + 1)}{2} \equiv 0 \pmod{11} \]

We see that

\[ (2r + 1)^2 + (2s + 1)^2 + (2t + 1)^2 + (u + 2)^2 \equiv 0 \pmod{11} \]
continued.

\[(2r + 1)^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}\]
\[(2s + 1)^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}\]
\[(2t + 1)^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}\]
\[(u + 2)^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}\]

This implies that there are multiple ways to make
\[(2r + 1)^2 + (2s + 1)^2 + (2t + 1)^2 + (u + 2)^2 \equiv 0 \pmod{11}\]
thus we can not proceed with this style of proof.
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Crank example

Thank you

References
1944 - Freeman Dyson, then an undergraduate at Cambridge, looks for a combinatorial proof of the congruences. To do this, he defines the rank $r$ of a partition $\pi$ as follows:

Let $\ell(\pi)$ be the largest part of $\pi$. Let $\nu(\pi)$ be the number of parts of $\pi$. Then

$$ r(\pi) := \ell(\pi) - \nu(\pi). $$
New Directions: Dyson

- 1944 - Freeman Dyson, then an undergraduate at Cambridge, looks for a combinatorial proof of the congruences. To do this, he defines the rank $r$ of a partition $\pi$ as follows:

Let $\ell(\pi)$ be the largest part of $\pi$. Let $\nu(\pi)$ be the number of parts of $\pi$. Then

$$ r(\pi) := \ell(\pi) - \nu(\pi). $$

- Rank can be easily visualized using Ferrers’ Diagrams.
Dyson also comes up with generating functions related to the rank, by defining \( N(m, n) \) as the number of partitions of \( n \) with rank \( m \), and \( N(m, q, n) \) as the number of partitions of \( n \) with a rank congruent to \( m \) (mod \( q \)).
Dyson also comes up with generating functions related to the rank, by defining \( N(m, n) \) as the number of partitions of \( n \) with rank \( m \), and \( N(m, q, n) \) as the number of partitions of \( n \) with a rank congruent to \( m \) (mod \( q \)).

Dyson claims that for \( n = 5k + 4 \),

\[
N(0, 5, 5k + 4) = N(1, 5, 5k + 4) = N(2, 5, 5k + 4) \\
= N(3, 5, 5k + 4) = N(4, 5, 5k + 4) \\
= \frac{p(5k + 4)}{5},
\]

and that a similar relationship exists for the mod 7 congruence.
Dyson also comes up with generating functions related to the rank, by defining $N(m, n)$ as the number of partitions of $n$ with rank $m$, and $N(m, q, n)$ as the number of partitions of $n$ with a rank congruent to $m \pmod{q}$.

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$$= N(3, 5, 5k + 4) = N(4, 5, 5k + 4)$$
$$= \frac{p(5k + 4)}{5},$$

and that a similar relationship exists for the mod 7 congruence.

He creates tables of calculated values to back up his claims, but does not provide a proof.
Let $n = 4$. $n = 5(0) + 4$. The partitions of $n$ are:
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- $1 + 1 + 1 + 1$: $r(\pi) = -3 \equiv 2 \pmod{5}$
- $2 + 1 + 1$: $r(\pi) = -1 \equiv 4 \pmod{5}$
- $2 + 2$: $r(\pi) = 0 \equiv 0 \pmod{5}$
- $3 + 1$: $r(\pi) = 1 \equiv 1 \pmod{5}$
- $4$: $r(\pi) = 3 \equiv 3 \pmod{5}$
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\[
4 : \quad r(\pi) = 3 \quad \equiv 3 \pmod{5}
\]

We have one partition in each group, modulo 5! Great!
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Exercise: Try this with $n = 5 = 7(0) + 5$. 
Let \( n = 6 \). \( n = 11(0) + 6 \). The partitions of \( n \) are:

\[
\begin{align*}
1 + 1 + 1 + 1 + 1 + 1 & : \quad r(\pi) = -5 \quad \equiv 6 \pmod{11} \\
2 + 1 + 1 + 1 + 1 & : \quad r(\pi) = -3 \quad \equiv 8 \pmod{11} \\
2 + 2 + 1 + 1 & : \quad r(\pi) = -2 \quad \equiv 9 \pmod{11} \\
2 + 2 + 2 & : \quad r(\pi) = -1 \quad \equiv 10 \pmod{11} \\
3 + 1 + 1 + 1 & : \quad r(\pi) = -1 \quad \equiv 10 \pmod{11} \\
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3 + 3 & : \quad r(\pi) = 1 \quad \equiv 1 \pmod{11} \\
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5 + 1 & : \quad r(\pi) = 3 \quad \equiv 3 \pmod{11} \\
6 & : \quad r(\pi) = 5 \quad \equiv 5 \pmod{11}
\end{align*}
\]
Rank example

Let $n = 6$. $n = 11(0) + 6$. The partitions of $n$ are:

- $1 + 1 + 1 + 1 + 1 + 1$: $r(\pi) = -5 \equiv 6 \pmod{11}$
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- $2 + 2 + 1 + 1$: $r(\pi) = -2 \equiv 9 \pmod{11}$
- $2 + 2 + 2$: $r(\pi) = -1 \equiv 10 \pmod{11}$
- $3 + 1 + 1 + 1$: $r(\pi) = -1 \equiv 10 \pmod{11}$
- $3 + 2 + 1$: $r(\pi) = 0 \equiv 0 \pmod{11}$
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- $4 + 1 + 1$: $r(\pi) = 1 \equiv 1 \pmod{11}$
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\end{align*}
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So rank doesn’t give us 11 equal groupings!
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Thank you

References
Once again, the mod 11 case proves to be difficult!
New Directions: Dyson

- Once again, the mod 11 case proves to be difficult!
- Dyson postulates that some other statistic of a partition exists that will divide the $5k + 4$, $7k + 5$, and $11k + 6$ partitions into equally-sized classes, and thus prove the congruences combinatorially. He names this statistic the “crank”, since he believes it is similar to the rank.
Once again, the mod 11 case proves to be difficult!

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- He leaves the finding of the crank, and the proof of his rank postulates to others...
1954 - A. O. L. Atkin and P. Swinnerton-Dyer prove Dyson’s rank postulate for the mod 5 and mod 7 congruences works. Several new congruences and partition identities come from their work.
1984 - Andrews and Garvan find the elusive “crank”!
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- Crank is similar to rank, as Dyson thought.
- Crank divides $p(5k + 4)$ into classes, based on its residue mod 5 - just like rank did.
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- Crank is similar to rank, as Dyson thought.
- Crank divides $p(5k + 4)$ into classes, based on its residue mod 5 - just like rank did.
- Andrews and Garvan prove that by using the crank, the classes of partitions are equal in size - for all three congruences!
Crank definition

Let \( \pi \) be a partition of \( n \).
Let \( \ell(\pi) \) be the largest part of \( \pi \).
Let \( \omega(\pi) \) be the number of ones in \( \pi \).
Let \( \mu(\pi) \) be the number of parts of \( \pi \) which are \( \geq \omega(\pi) \).
Crank definition

Let $\pi$ be a partition of $n$.
Let $\ell(\pi)$ be the largest part of $\pi$.
Let $\omega(\pi)$ be the number of ones in $\pi$.
Let $\mu(\pi)$ be the number of parts of $\pi$ which are $> \omega(\pi)$.

Then the crank is defined as

$$c(\pi) = \begin{cases} 
\ell(\pi), & \text{if } \omega(\pi) = 0 \\
\mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0
\end{cases}$$
Let $n = 4$. $n = 5(0) + 4$. The partitions of $n$ are:
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- $1 + 1 + 1 + 1$: $c(\pi) = -4 \equiv 1 \pmod{5}$
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\end{align*}
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So crank gives us one partition in each group, modulo 5, just like rank did. So far, so good.
Crank example

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So crank works for the troublesome mod 11 congruence!
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4 + 2 & : \quad c(\pi) = 4 \quad \equiv 4 \pmod{11} \\
5 + 1 & : \quad c(\pi) = 0 \quad \equiv 0 \pmod{11} \\
6 & : \quad c(\pi) = 6 \quad \equiv 6 \pmod{11}
\end{align*}
\]

So crank works for the troublesome mod 11 congruence!
Q: Why do motorcycle gang members use their motorcycles to get to work?
Q: Why do motorcycle gang members use their motorcycles to get to work?
A: Because members of cyclical groups commute.
Thank you for listening.
Thanks for a fun quarter, Stephanie!!
Have a great break.
References